On the Link between Piketty's Laws

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Thomas Piketty has drawn worldwide attention with his proposition that the disparity between workers and capitalists is increasing, and that governments should intervene to bring this process to a standstill. In addition to his thesis being an interesting and novel one, the popularity of Piketty's book, *Capital in the 21st Century* (2014), is also due to the publication of his comprehensive dataset and the many resources behind those data. A final reason for his popularity is the relative simplicity of the main formulas in the book, which he named the 'Fundamental Laws of Capitalism'. The first law applies to the capital share in income (α) and the second law to the capital coefficient (β). These simple formulas have their limitations, one of which is crucial and I will highlight here. In a column for Voxeu (van Schaik 2014), I have shown what will happen with Piketty's laws when investment replacement and depreciation is added to these laws, assuming a fixed technical lifetime of capital goods. Below I extend this analysis to the endogenous determination of real wages, which also endogenizes the lifetime of capital goods. This approach brings forward an alternative view on the link between Piketty's laws. References to the relevant pages in the book of Piketty (2014) are in parentheses.

The main thesis

Piketty's main thesis is that wealth has been growing faster than income since 1970. This thesis is based on the observation that the growth rate of income g in the nine countries surveyed is decreasing, while the rate of profit r hardly changes. Besides, Piketty observes a simultaneous long-term rise in the wealth/income ratio and the capital share in income. Piketty (2014, 220–221) expects that in the 21st century these developments will continue, because

> "over a very long period of time, the elasticity between capital and labour seems to have been greater than one: an increase in the capital/income ratio β seems to have led to a slight increase in α, capital's share of national income, and *vice versa.* Intuitively, this corresponds to a situation in which there are many different uses for capital in the long run. Indeed, the observed historical evolutions suggest that it is always possible – up to a certain point, at least – to find new and useful things to do with capital: for example, new ways of building and equipping houses (think of solar panels on rooftops or digital lighting controls), ever more sophisticated robots and other electronic devices, and medical technologies requiring larger and larger capital investments".

To explain the simultaneous long-term rise in the capital/income ratio β and the capital share of income α Piketty reverts to the neoclassical model of distribution of income by relying on a high value of the elasticity of substitution between capital and labour ($\sigma > 1$). In this model the rate of profit is related to the marginal product of capital. Using a CES production function the relation between α and β is derived as

$$(1)\quad \alpha = a\beta^{\frac{\sigma-1}{\sigma}} \rightarrow \Delta \ell n \alpha = \frac{\sigma - 1}{\sigma} \Delta \ell n \beta$$

The capital share in income is an increasing function of the capital coefficient if, and only if, the elasticity σ is greater than 1. If σ gets closer to 1, then the CES function tends to a Cobb-Douglas function with a constant distribution of income between workers and capitalists. If $\sigma \leq 1$ the simultaneous increase of α and β will not hold.

Piketty (2014, 221) claims that, on the basis of historical data, one can infer that σ lies between 1.3 and 1.6. This claim has been criticised across the board, mainly because Piketty introduces a more encompassing definition of wealth than conventional measures of capital, which typically yields values of σ that are much lower than 1 (see Rowthorn 2014; Acemoglu and

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Table 1

Decomposition of national wealth into four components (1810–2010)

	Housing wealth			Agricultural land			Other domestic capital			Net foreign wealth		
	USA	France	UK	USA	France	UK	USA	France	UK	USA	France	UK
1810	22	21	16	42	51	45	42	28	39	– 6	1	0
1870	22	25	18	23	34	26	59	27	42	– 4	14	14
1910	22	25	21	17	20	5	64	39	50	– 3	16	25
1950	38	28	36	5	15	7	55	55	57	1	1	0
1970	38	34	33	5	12	3	56	51	62	1	4	2
2010	42	61	57	3	2	1	61	39	46	– 6	– 2	– 4

Note: Decennial averages (%), except 1910 and 2010.

Sources: Piketty and Zucman (2013), Tables A18, A21, A24, A27.

Robinson 2014). This raises the question of whether the neoclassical model of income distribution is an adequate vehicle to handle the simultaneous long-term rise in the wealth/income ratio and the capital share of income.

National wealth consists of four components, housing wealth, agricultural land, other domestic capital and net foreign wealth. Table 1 decomposes national wealth into these four components over two centuries (1810–2010) for three countries with data already available from 1810 (United States, France and Britain). The table illustrates important historical tendencies. Over the course of time the share of agricultural land has decreased to a minimum. In France and Britain net foreign wealth was high during the first wave of globalization (1870–1910). The share of 'other domestic capital' in France and Britain decreased substantially after the 1970s. And most importantly, the share of housing wealth rose instantaneously in both of the centuries under consideration (although not included in the table, the share of housing wealth in Germany climbed from 28 percent in 1950 to 57 percent in 2010). This pattern may well continue, in which case Piketty could be right about the rise of capital in the 21st century.

Evidently, not only housing, but also structures contribute to national wealth. As Rognlie (2014) observes, structures continue to comprise the vast majority of the private capital stock in the United States: 175 percent of GDP, as compared to, for instance, 'information processing equipment' (computers, communication, medical, etc.), which represent only 8 percent of GDP. As a result, the largest share of capital consists of housing wealth and structures, which depreciate less than equipment.

In a column (van Schaik 2014), I have shown what happens with Piketty's laws when investment replacement and depreciation is added to them, assuming a fixed technical lifetime of capital goods. In the column I introduced the so-called reproduction model, which goes back to von Neumann and Sraffa (see Schefold 1980). Here I extend this analysis to the endogenous determination of real wages, which also endogenizes the lifetime of capital goods.

The model

In *Reproduction and Fixed Capital* (van Schaik 1976a) I showed that the clay-clay vintage model can be regarded as a special case of the reproduction model with fixed capital of unequal efficiency. In this model, in the steady state, the capital coefficient is uniform for all vintages and there is only labour-augmenting technological progress, embodied in new investment. In other words, labour requirements per unit of equipment decrease at a constant rate the younger the vintage is. Labour requirements become fixed at the moment of the installation of new capital goods.

The model implies that *direct* substitutability between labour and capital is virtually non-existent. This applies to old as well as new vintages of capital. From a macroeconomic point of view, however, substitutability between labour and capital does exist *indirectly*. This comes about through replacement of capital of the oldest vintage in use with relatively high labour-output ratio by new investment with lower labour-output ratio due to labour-augmenting technological progress. Though labour productivity of old vintages does not improve during the lifetime of those vintages, replacement of the oldest vintages by new investment does bring about a sustained growth in output per worker. The economic lifetime of the capital stock is determined by the equality of the real wage rate and output per worker of the oldest vintage in use, the so-called marginal vintage.

Table 2

Replacement rate (d) and labour share in income (λ)

Economic lifespan	g = 0.04		g = 0.02	
θ	D	λ = dθ	d	λ = dθ
10	0.0833	0.83	0.0913	0.91
20	0.0336	0.67	0.0412	0.82
30	0.0178	0.54	0.0246	0.74

Note: The replacement rate is calculated as = $\frac{g}{(1+g)^{\theta}-1}$, which is the fraction of the capital stock replaced every period in a balanced growth path with exogenous lifetime of capital (θ) and growth rate (g).

The model is described in Appendix A.[1] The model consists of two sub-systems: a quantity system and a price system, which together determine the economic lifetime. The quantity system is the von Neumann side of the model and the price system is linked to the name of Sraffa. To keep the model tractable, population growth is assumed to be zero, implying that the (net) rate of growth equals the rate of technological progress, which – following Piketty – is exogenous to the model.

The quantity system yields the solution for the replacement rate as a function of the growth rate and the economic lifespan. The rate at which capital is replaced is endogenous in the model. Along a balanced growth path with a given growth rate it is one-to-one related to the economic lifetime of capital. Table 2 gives a numerical example of the values of the replacement rate for different values of the growth rate and the economic lifespan. The example shows that, for a given economic lifetime, the replacement rate increases as the growth rate is lower. This result was described earlier in van Schaik (2014).

Table 2 also demonstrates the sensitivity of the labour share of income for changes in the values of the growth rate and the economic lifespan. By definition, the labour share is the ratio of the real wage rate and macroeconomic labour productivity. In equilibrium, the real wage rate equals the labour productivity of the marginal vintage. As derived in Appendix A, the labour share can be expressed as the product of the economic lifetime and the replacement rate λ = dθ. Interestingly, Table 2 shows that, for a given lifetime, the labour share is higher as the growth rate decreases. By contrast, for a given growth rate, the labour share decreases as the economic lifetime increases. The latter is explained by the slowdown in real wages, induced by the lower labour productivity of the marginal vintage.

The figures in Table 2 point to the tendency of the labour share to approach 100 percent, if the economic lifetime is held constant and the growth rate is lower. However, in the complete model, the economic lifetime is endogenous and not only depends on the rate of growth, but also on the rate of profit (rate of return to capital), so that the price system has to be taken into account. The price system describes the capital returns on all vintages in use. Following Sraffa (1960), in Appendix A I have shown that in equilibrium the gross rate of profit on each vintage is an annuity, which can be decomposed into the depreciation rate δ and the net rate of profit r. The depreciation rate is a function of the net rate of profit and the economic lifetime.[2]

As all vintages earn the same rate of gross profit, the price system can be reduced to a core relation describing the distribution of income between capitalists and workers. Piketty's *first* law is the definition of the capital share in income, which now includes depreciation α = (δ + r)β. For a given capital coefficient, equating the labour share from the quantity system with the labour share of the price system yields the economic lifetime

$$(2)\ \ d\theta = 1 - (\delta + r)\beta \rightarrow \theta = \frac{1-(\delta+r)\beta}{d}$$

The full solution of the model is obtained by invoking Piketty's second law, which determines the value of the capital coefficient and is shown in the next section.

Effects of changes in the r-g gap

The solution of the complete model provides the long-term situation of balanced growth. Table 3 contains some numerical examples.

The table features four growth paths – the first with a growth rate of 5 percent, and the last with 2 percent growth. These are net growth rates. The sum of the replacement rate and the net rate of growth is the gross rate of growth. Piketty assumes a given net macroeco-

[1] In Appendix A the term L measures the effective number of workers. It takes into account the number of workers and the efficiency of each worker. The ratio between output Y en L is output per effective worker, which in the steady state is constant. In the steady state output per worker (labour productivity) grows with the rate of labour-augmenting technological progress g.

[2] Note that in van Schaik (2014), the depreciation rate did not change, because the lifetime of capital goods was fixed.

Table 3

Effects of a larger r-g gap (exogenous net rate of profit)

Net rate of growth g (exogenous)	5	4	3	2
Gross savings rate s (exogenous)	20.5	20.5	20.5	20.6
Economic life span θ (A12)	10	13	19	38
Replacement rate d (A4)	8.0	6.0	4.0	1.8
Gross rate of growth d+g	13.0	10.0	7.0	3.8
Capital coefficient β (Piketty's Second Law of Capitalism) (A10)	1.6	2.1	2.9	5.5
Net rate of profit r (exogenous)	5	5	5	5
Depreciation rate δ (A8)	8.0	5.7	3.3	0.9
Gross rate of profit δ+r	13.0	10.7	8.3	5.9
Capital share in income α (Piketty's First Law of Capitalism) (A9)	0.205	0.22	0.24	0.32
Labour share in income λ (A7)	0.795	0.78	0.76	0.68
Capital stock K	158	205	294	545
Output Y	100	100	100	100
Net investment gK	7.9	8.2	8.8	10.9
Replacement investment dK	12.6	12.3	11.7	9.7
Gross investment	20.5	20.5	20.5	20.6
Net profits rK	7.9	10.3	14.7	27.2
Depreciation δK	12.6	11.6	9.6	5.1
Gross Profits	20.5	21.9	24.3	32.3
Profit surplus (profits – investment)	0	1.4	3.8	11.7

Note: The number in parenthesis refers to the equations in Appendix A. The economic lifetime is chosen to be an integer, so that the gross savings rate is a residual in the calculations.

nomic saving rate (Piketty and Zucman 2013, 1272). Here, I will assume a given gross savings rate s, which now also includes depreciation. In long term equilibrium, gross savings sY equal break-even investment (d+g)K (Mankiw 2007, 207). At this equilibrium the capital coefficient does not change anymore. This leads to Piketty's *second* law, now including replacement investment

(3) $\beta = \frac{s}{d+g}$

Assuming a gross savings rate of about 20 percent (Piketty 2014, 178), the capital coefficient increases from 158 percent in the case of a net rate of growth of 5 percent to 545 percent with a net rate of growth of 2 percent. This represents a formidable rise of the ratio between capital and income, and the strength of Piketty's argument is that he also finds this increase in his data for the period 1970-2010 (Piketty 2014, 26). The increase in the capital coefficient happens during the Traverse to the new steady state with a lower rate of growth. In the (new) steady state K and Y have the same growth rate g, so that the capital coefficient remains fixed.

The capital stock is a fixed percentage of income along the balanced growth path. Therefore, replacement investment is also a percentage of income. At high growth rates the economic lifetime is short, so that replacement investment is much higher than net investment. Thus, a considerable of part of savings is allocated to replace the capital stock, and not to the expansion of the capital stock.

Contrary to the partial analysis in Table 2, it now appears that the labour share in income decreases if the economy is stuck on a path with lower economic growth. The reason is that the lower growth rate induces a longer economic lifetime. The increase in the number of vintages is made possible by the adaptation of real wages to the lower labour productivity of the marginal vintage.[3]

As stated above, Piketty's *first* law (Piketty 2014, 52) is the definition of the capital income ratio α = (δ + r)β. According to Piketty (2014, 202) the net rate of profit is on average at the 5-percent level, although lower values are not excluded in the future. In Table 3 the rate of profit is 5 percent at every growth path.

The sum of the depreciation rate and the net rate of profit is the gross rate of profit. At lower rates of growth the capital coefficient increases, which has a positive effect on the capital share. On the other hand, the gross rate of profit decreases as growth is lower. This is due to the decrease in the depreciation rate, in-

[3] Note that during the Traverse from one steady state to another steady state, the development of real wages is governed by the institutional environment of the economy (Acemoglu and Robinson 2014). The Netherlands is a typical example of a country where institutions have embraced wage moderation to cope with the forcing up of the productivity slowdown on the labour share of income. The early introduction in the 1970s of the clay-clay vintage model in macroeconomic policy analysis by the CPB Netherlands Bureau for Economic Policy Analysis played an important role in shaping this consensus. One of the first estimations of this model is Hartog and Tjan (1976) of CPB.

duced by the lengthening of the economic life span. By contrast, the capital share increases from 20.5 percent on the growth path of 5 percent to more than 32 percent on the growth path of 2 percent. The reason for this is that the decrease in the gross rate of profit is smaller than the increase in the capital coefficient. In Table 3, comparing the growth path of 2 percent with the growth path of 3 percent, the gross rate of profit decreases by 33 percent, whereas the capital coefficient rises by 62 percent. This shows that the vintage model with only *indirect* substitution between labour and capital is fully able to explain the simultaneous long-term rise in the wealth/income ratio and the capital share in income. So, to this end, there is no need to resort to the neoclassical model by assuming an elasticity of substitution that exceeds 1.

Table 3 also shows that both net and gross profits rise as growth is lower. In each steady state, macroeconomic savings equal about 20.5 percent of the income. The savings rate exactly equals the capital share in income if the growth rate is equal to the rate of profit: r = g. This is the situation in the Golden Age of the 1950s and 1960s, where r and g amounted to approximately 5 percent.[4] In the decades that followed, growth slowed down, while profit rates remained high. This is why the surplus of profits rose. With a growth rate of 2 percent, this surplus represents almost 12 percent of income. This explains, for example, why countries that have experienced a marked slowdown in economic growth, such as Germany and Japan, have been confronted with huge surpluses in the current account of the balance of payment, and with a huge rise in net foreign wealth as a result.[5]

The mirror image of the rise in surplus profits is the decrease of capitalists' savings rate. In Table 3, where the net rate of profit is given, this savings rate remains implicit. In practice both workers and capitalists save. However, as shown in Appendix B, under certain assumptions, workers' savings rate drops out of all equilibrium relations and capitalists' savings rate remains the only relevant saving propensity

$$(4)\quad (\delta + r) = \frac{1}{s_c}(d + g)$$

This is the Cambridge equation, developed by Luigi Pasinetti (1962). The equilibrium rate of profit emerges as being determined by the rate of growth divided by the capitalists' propensity to save. For a given economic lifetime the growth rate is also fixed, so that an increase in capitalists' saving rate will lead to a lower rate of profit. Table 4 gives an example, describing a

Table 4

The effects of a smaller r-g gap (endogenous net rate of profit)

Net rate of growth g (exogenous)	2	2
Gross savings rate s (exogenous)	0.206	0.206
Economic life span θ (A12)	38	31
Replacement rate d (A4)	1.8	2.4
Gross rate of growth d+g	3.8	4.4
Capital coefficient (Piketty's Second Law of Capitalism) (A10)	5.5	4.7
Capitalists' saving rate (exogenous)	0.638	0.767
Net rate of profit r (endogenous)	5	4
Depreciation rate δ (A8)	0.9	1.7
Gross rate of profit δ+r	5.9	5.7
Capital share in income (Piketty's First Law of Capitalism) (A9)	0.32	0.27
Labour share in income λ (A7)	0.68	0.73
Capital stock K	545	472
Output Y	100	100
Net investment gK	10.9	9.4
Replacement investment dK	9.7	11.1
Gross investment	20.6	20.6
Net profits rK	27.2	18.9
Depreciation δK	5.1	8.0
Gross Profits	32.3	26.9
Profit surplus (profits – investment)	11.7	6.3

Note: The numbers in parentheses refer to the equations in Appendix A. The gross rate of profits follows from the Cambridge equation in Appendix B. The net rate of profit is chosen to be an integer, so that capitalists' savings rate is a residual in the calculations.

[4] In the reproduction model, the equality of r and g leads to the Golden Rule of Accumulation whereby consumption per worker is at its greatest level (see van Schaik 1976a, Appendix C). In this case, the capital share in income equals the savings rate.

[5] In 2010 net foreign wealth accounted for 9 percent of national wealth in Germany and 11 percent in Japan (Piketty and Zucman 2013, Table A27).

steady state with a net growth rate of 2 percent. The first column corresponds to the last column of Table 3.

Assuming that capitalists increase their saving rates, the exercise in Table 4 shows that the higher savings rate leads to a decline in the gross rate of profit. According to the Cambridge equation, this is the outcome of two opposite forces, a positive effect of the increase in the gross rate of growth and a negative effect of the higher capitalists' saving rate. The lower rate of profit is accompanied by a shorter economic lifetime, which raises the replacement rate and consequently the gross rate of growth. According to Piketty's second law, the higher rate of growth lowers the capital coefficient. This has a substantial effect on the distribution of income between workers and capitalists. The capital share in income decreases from 32 to 27 percent, because both the gross rate of profit and the capital coefficient decline, so that the negative effect on the capital share is unambiguous. It is important to note that a further increase in the capitalists' savings rate will further lower the disparity between workers and capitalists, although the wealth/income ratios will not completely return to the relatively low values that correspond with high growth rates.

Conclusion

In the last section of 'Capital Is Back' (2013), Piketty and Zucman take a brief look at the implications of their new data on capital for understanding the evolution of factor shares and the shape of the production function. The data from 1975 onwards show that capital shares increased in all rich countries from about 15–25 percent in the 1970s to 25–35 percent in 2010.[6] By their estimates, capital coefficients have risen even more than capital shares, so that rate of profit has declined somewhat. This is what one would expect in any model: when there is more capital, the rate of profit must go down. However, according to the data, the rate of profit has fallen less than the quantity of capital, implying a rising capital share. Piketty and Zucman (2013, 1303) say there are several ways to think about this piece of evidence.

> "One can think of a model with imperfect competition and an increase in the bargaining power of capital (e.g. due to globalization and increasing capital mobility). A production function with three factors – capital and high-skill and low-skill labour– where capital is more strongly complementary with skilled than with unskilled labour would also do, if there is a rise in skills or skill-biased technical change. Yet another – and more parsimonious – way to explain the rise in α is a standard two-factor CES production function F(K,L) with an elasticity of substitution $\sigma > 1$".

[6] Here the capital shares are net of depreciation.

The last explanation has been criticised across the board, because conventional measures of capital typically yield values of σ that are much lower than 1. To discover other possibilities – in addition to the implications mentioned above – I have brought a clay-clay vintage model in a special case of joint production to the forefront. Within this framework, substitutability between labour and capital only exists indirectly. This comes about through replacement of capital of the oldest vintage in use with relatively high labour-output ratio by new investment with lower labour-output ratio due to labour-augmenting technological progress. The analysis reveals that this reproduction model is well able to explain the simultaneous long-term rise in the wealth/income ratio and the capital share in income.

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Appendix A: the clay-clay vintage model in special case of joint production[7]

Quantity system

Consider a closed economy $Y = C + S$ with a given savings rate $s = S/Y$. Here Y is output. The economic lifetime of fixed capital is denoted by θ. The capital coefficient is β, whereas the constant rate of labour-augmenting technical progress is g. Population growth is zero. Assuming that old capital goods are not regarded as net output, the circular flow $Y = C + I$ can be decomposed into

(A1) $$y^o + y^1 + \cdots + y^{\theta-1} = C + (1+g)\beta y^o$$

In the steady state output of vintages in use is

(A2) $$y^1 = \frac{y^o}{(1+g)}, \quad y^2 = \frac{y^o}{(1+g)^2}, \dots \quad \dots, y^{\theta-1} = \frac{y^o}{(1+g)^{\theta-1}}$$

Macroeconomic output is the sum of output of vintages in use

$$Y = y^o\left[1 + \frac{1}{(1+g)} + \cdots + \frac{1}{(1+g)^{\theta-1}}\right] \rightarrow$$

$$Y = y^o \frac{(1+g)^\theta - 1}{g(1+g)^{\theta-1}} \rightarrow$$

(A3) $$\frac{g(1+g)^\theta}{(1+g)^\theta - 1} Y = (1+g)y^o = (d+g)Y$$

The ratio in the LHS is the gross rate of growth, which can be decomposed into the net rate of growth g and the replacement rate d

(A4) $$d = \frac{g}{(1+g)^\theta - 1}$$

As the macroeconomic capital stock is βY, gross investment can be written in two ways

$$(d+g)\beta Y = (1+g)\beta y^o$$

Effective labour is

$$L = (1+g)ay^o + (1+g)^2 ay^1 + \cdots + (1+g)^\theta ay^{\theta-1}$$

Using (A2), this equation reduces to

$$L = a\theta(1+g)y^o = a\theta(d+g)Y$$

Output per effective worker is

(A5) $$\frac{Y}{L} = \frac{1}{a\theta(d+g)}$$

Output per effective worker of the oldest vintage in use, the so-called marginal vintage is

(A6) $$\mu = \frac{1}{a(1+g)^\theta}$$

By definition, the labour share in income is the ratio of real wage and output per worker. In equilibrium the real wage of an effective worker equals output per effective worker of the marginal vintage, so that the labour share can be written as

(A7) $$\lambda = \frac{\frac{w}{p^o}}{\frac{Y}{L}} = \frac{\theta(d+g)}{(1+g)^\theta} = d\theta$$

Notice that neither the capital coefficient nor the labour coefficient of the newest vintage plays a role in this expression.[8]

Price system

Capital return is the sum of depreciation net rate of profit

$$(p^o - p^1)\beta + rp^o\beta$$
$$(p^1 - p^2)\beta + rp^1\beta$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$(p^{\theta-1} - 0)\beta + rp^{\theta-1}\beta$$

Dividing by the price of total output and rearranging terms, capital returns are[9]

$$(1+r)\frac{p^o}{p^o}\beta - \frac{p^1}{p^o}\beta$$
$$(1+r)\frac{p^1}{p^o}\beta - \frac{p^2}{p^o}\beta$$
$$\dots\dots\dots\dots\dots\dots$$
$$(1+r)\frac{p^{\theta-1}}{p^o}\beta$$

Relative book values only depend on the rate of profit and the life span of capital goods (see Sraffa 1972 and van Schaik 1976a).

$$\frac{p^t}{p^o} = \frac{(1+r)^\theta - (1+r)^t}{(1+r)^\theta - 1}, \quad t = 1, 2, \dots, \theta - 1$$

An example is the second vintage

$$(1+r)\frac{(1+r)^\theta - (1+r)}{(1+r)^\theta - 1} - \frac{(1+r)^\theta - (1+r)^2}{(1+r)^\theta - 1} = \frac{r(1+r)^\theta}{(1+r)^\theta - 1}$$

[7] This appendix is based on van Schaik (1976a).

[8] Empirical applications of the clay-clay vintage model always estimate the Traverse from one potential steady state to another potential steady state. An example is the break by WWII. The parameters β, a, s and g were (much) lower in the pre-war period than afterwards (see van Schaik 1976b).

[9] Notice that $p^o = 1$ (Numéraire).

This annuity is the gross rate of profit, which can be decomposed into the net rate of profit r and the depreciation rate δ

$$\text{(A8)}\quad \delta = \frac{r}{(1+r)^{\theta}-1}$$

All vintages earn the same gross rate of profit. As a result, each vintage describes the macroeconomic distribution of income between capitalists and workers. Piketty's 1st law is the definition of the capital share in income, which now includes depreciation

$$\text{(A9)}\quad \alpha = (\delta + r)\beta$$

According to Piketty's 2nd law the capital coefficient is the ratio of savings rate and the rate of growth, which now includes replacement investment

$$\text{(A10)}\quad \beta = \frac{s}{d+g}$$

Using (A9) and (A10) the labour share in income is

$$\text{(A11)}\quad \lambda = 1-\alpha = 1-(\delta+r)\beta = 1-\frac{(\delta+r)}{(d+g)}s$$

Equating the labour share from quantity system (A6) with the labour share from the price system (A9) yields the economic life span θ

$$\text{(A12)}\quad d\theta = 1-\frac{(\delta+r)}{(d+g)}s$$

Appendix B: the Cambridge equation

Following Pasinetti (1962), the irrelevance of the workers' saving rate can be explained as follows. If both capitalists and workers have a positive saving rate, in the long run, the capital stock owned by each category of savers becomes proportional to their savings, so that (using subscript c for capitalists and w for workers)

$$\text{(B1)}\quad \frac{S}{K} = \frac{S_c}{K_c} = \frac{S_w}{K_w}$$

But profits are also proportional to the capital stocks. If workers lend their capital to capitalists and the assumption is made that the rate of interest on loans is equal to the rate of profit, then the rate of profit is

$$\text{(B2)}\quad (\delta+r) = \frac{P}{K} = \frac{P_c}{K_c} = \frac{P_w}{K_w}$$

Dividing through yields

$$\text{(B3)}\quad (\delta+r)\frac{K}{S} = \frac{P}{K}\frac{K}{S} = \frac{P_c}{S_c} = \frac{P_w}{S_w}$$

Since in equilibrium S=I (W = labour income)

$$\text{(B4)}\quad (\delta+r)\frac{K}{I} = \frac{P}{I} = \frac{P_c}{s_c P_c} = \frac{P_w}{s_w(W+P_w)}$$

From the last equality it follows

$$\frac{s_c P_w}{s_w(W+P_w)} = 1 \rightarrow s_c P_w = s_w(W+P_w)$$

This shows that, in all equilibrium growth relations, workers' savings always become equal to – and hence can be replaced by – the amount of savings the capitalists would make if workers' profits were to go to them. Hence the workers' savings rate drops out of all equilibrium relations and the capitalists' savings rate remains the only relevant saving propensity.

Multiplying (B4) by the rate of growth I/K

$$(\delta+r) = \frac{P}{K} = \frac{1}{s_c}\frac{I}{K} = \frac{1}{s_c}(d+g)$$

This is the Cambridge equation. The equilibrium rate of profit emerges as being determined by the rate of growth divided by the capitalists' propensity to save.