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Abstract

This paper extends the Hartman model to study the optimal rotation age of two interdependent stands when the stream of amenities produces from the two stands may be complements or substitutes, both in space and over time. In the presence of stand interdependence both the ownership structure and the sequence of decision making matters. Rotation age choices are examined and compared under a variety of equilibria, including Nash, Stackelberg, and sole owner cases which differ as to the level of commitment by landowners to their choices. We show that the sole owner's rotation age is longer than the rotation age solved under both Nash and Stackelberg assumptions if the stands are spatial complements, but shorter if they are substitutes. The precise relationship between the Nash and Stackelberg rotation ages, and the qualitative properties of rotation ages in terms of timber prices, regeneration costs, and interest rates, also depend on how spatial substitutability and complementarity between stands evolves through time.

JEL Classification: Q23, H21.

Keywords: forest rotation, amenity services, stand interdependence.

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1. Introduction

Forest ecosystems comprise complex site-specific interactions between plant and animal species. One aspect of forest ecosystems rarely acknowledged in economics models is the notion of stand interdependence, rather, the majority of models consider management of only a single stand in isolation. In practice, the management of each stand in a given region should not be undertaken independently of other stands. Biologists have long known this, arguing that trees of many age classes and species mixes are necessary for conservation of biodiversity or contiguous habitat for certain animal species. Stand interdependence may also be anthropogenic in nature. For instance, the recreational opportunities of larger forest areas may be dependent on the interaction or coordinated management of several stands.

Managing the interdependent multiple-stand forest is a challenging task. Harvesting even one stand may sometimes pose a threat to the maintenance of an entire ecosystem. While the task is difficult enough for one manager, it becomes even more difficult under the reality of nonindustrial private landownership. Land property rights usually do not follow forest cover types. This means management of adjacent stands is likely not coordinated among landowners. At the extreme, landowners owning one piece of an ecosystem will neglect, knowingly or unknowingly, the impact of their private harvesting on the whole ecosystem or on other nearby landowners.

The behavior of landowners who have no incentives to coordinate actions will be socially costly. In fact, the impact of one landowner's decisions on the forest ecosystem used by another landowner represents a type of economic externality associated with private forest management. Only a social planner who manages the entire forest ecosystem has incentives to solve for the rotation age of each stand, conditional on its impacts to all other stands.

In this paper we examine several issues not addressed in the economic management of interdependent forest stands. We first review the concept of interdependent stands as it appears in the literature. This amounts to having timber and non-timber amenity benefits which depend on rotation ages of an adjacent stand. We then examine various assumptions for the timing of adjacent landowner decisions. The timing of decisions follows from each landowner's ability to credibly commit to a harvesting action. If landowners are unable to commit, then they effectively make forest management decisions simultaneously, i.e., landowners play a Nash game. The Nash equilibrium reflects private ownership in practice, where landowners typically border a small number of neighbouring landowners and make decisions without regard to the other landowners. An alternative setting is examined where one landowner commits to moving first, making decisions with the reaction of another landowner in mind. Finally, we examine the rotation age decision for a sole owner who makes forest management decisions taking into account the interdependence between *all* stands. Comparison of this outcome with the simultaneous and first-mover outcomes will show the importance of coordination, and thereby hint at the social cost of not coordinating forest management actions.

There are very few analytical treatments of the economics problem behind stand interdependence. Stand interdependence was originally discussed in Bowes and Krutilla (1985, 1989), who proposed a linear programming approach to maximize the rents associated with multiple stands under a single (government) owner. Swallow and Wear (1993) and Swallow et al. (1997) were the first to formulate explicit spatial interactions for non-timber amenity benefits between two adjacent stands, but their analysis relies on numerical approximations. Koskela and Ollikainen (2001b) examined the rotation age decision for a single landowner making decisions for a single stand, under the assumption of a purely exogenous adjacent stand. Their work does not focus on the different landowner commitment assumptions that we examine, nor do they examine the important sole owner outcome. All of these issues are critical to understanding the private landowner case.

There is a large literature on stand interdependence in other settings, such as species conservation. This work, when taken in the context of forested areas, promotes the idea that multiple stands are needed to sustain certain species (see e.g. Csuti et al. 1997, Ando et al. 1998, Polasky et al. 2000). Also, an increasing number of empirical studies on conservation, ecosystem management, and forest management exists, but these are typically undertaken only from the viewpoint of a benevolent social planner (see e.g. Beavers et al. 1995, Albers 1996, Beavers and Hopf 1999, Haight and Travis 1997, Montgomery 1998). Unlike our paper, this literature either considers only the case of the sole owner, or it is based on site-specific empirical data. Hence it too abstracts from the interesting practical problems that follow from private landowners with no incentives to coordinate management of their forests.

The rest of our paper is organized as follows. In section 2 we introduce an extended Hartman model of forest management and make specific the definition of spatial dependence between stands and its evolution over time, i.e. temporal dependence. We then analyse rotation age under simultaneous move, first-mover and sole owner timing assumptions. Section 3 characterizes the qualitative dependence of rotation ages on important parameters. Finally, in section 4 we provide some concluding remarks.

2. A model of interdependent stands

We first describe a basic framework for the determination of rotation ages for two adjacent stands, denoted by stand $'a'$ and stand $'b'$. It is assumed that landowners value net harvest revenue and the non-timber amenity services produced from the stand, just as in the conventional Hartman model of forest management (Hartman 1976). Following Swallow and Wear (1993) and Koskela and Ollikainen (2001b), we assume that stands are interdependent in terms of amenities, but independent with regard to timber production.

The growth of stands *a* and *b* is an S-shaped function of rotation age. Timber volume at harvest is denoted by $f(T)$ and $g(\tau)$, where T refers to the rotation age for stand *a* and τ refers to the rotation age of the stand *b*. Timber prices p and q and regeneration costs c_T and c_τ for stand *a* and *b,* respectively, are allowed to differ between the stands. These assumptions reflect the typical situation in which stands differ inherently due to site characteristics (such as slope, tree species, aspect, or access). Prices, costs, and the real interest rate r are assumed to be constant over time, as with the basic Hartman model. The present values of timber production over an infinite cycle of rotations for each stand are, respectively,

$$V^a = (1-e^{-rT})^{-1}\left[pf(T)e^{-rT} - c_T\right] \tag{1a}$$

$$V^b = (1-e^{-r\tau})^{-1}\left[qg(\tau)e^{-r\tau} - c_\tau\right]. \tag{1b}$$

We now introduce amenity values in a manner that reflects stand interdependence. Let $F^a(s,\tau)$ describe valuation of amenity benefits provided by stand *a* at *time* s when the adjacent stand *b* has a *rotation age* of τ . Likewise, $F^b(T,x)$ denotes valuation of amenity benefits of stand *b* at *time* x when stand *a* has a *rotation age* of T. Using this notation, the present value of amenities over an infinite series of rotations of length T and τ for both stands are written,

$$E^a = (1-e^{-rT})^{-1} \int_0^T F^a(s,\tau)e^{-rs}ds \tag{2a}$$

$$E^b = (1-e^{-r\tau})^{-1} \int_0^\tau F^b(T,x)e^{-rx}dx . \tag{2b}$$

For subsequent analysis we must also characterize how the amenity values in equations (2a) and (2b) behave in terms of changes in their own rotation age and changes in the adjacent stand's rotation age. In describing these effects, we will use the label *own-stand* to refer to the stand in question, and *adjacent stand* to refer to the other stand.

Neglecting for a moment the present value terms in (2a) and (2b), and differentiating the integrals for each stand with respect to the own rotation age, we can obtain a marginal amenity valuation function defined at harvest times for both stand *a* and *b*: $F^a(T,\tau)$ and $F^b(T,\tau)$, respectively.

We now present two definitions from the literature which characterize *spatial dependence* of stands, as well as the evolution of this over time, i.e., *temporal dependence*. Differentiating each stand's marginal amenity valuation function with respect to the rotation age of the *adjacent* stand indicates how the marginal amenity valuation changes with respect to changes in the rotation age of the other stand. These derivatives are termed spatial dependence in the literature and are summarized in:[1]

Definition 1 (Koskela and Ollikainen 2001b). S*patial Dependence*

$$F^a_\tau(T,\tau) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0 \text{ and } F^b_T(T,\tau) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0 \text{ if stands are } \begin{cases} \text{substitutes wrt amenities} \\ \text{independents wrt amenities} \\ \text{complements wrt amenities} \end{cases}$$

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[1] In what follows, derivatives of functions will be denoted by subscripts unless otherwise noted.

Definition 1 is consistent with ALEP complementarity/substitutability first formalized by Samuelson (1974) and others in a different context.[2] If the stands are spatial substitutes, then the marginal amenity valuation of each stand decreases with the rotation age of the adjacent stand. If the stands are spatial complements, then the opposite is true, i.e., marginal amenities of each stand increase with the rotation age of the adjacent stand.

It is also important to know how spatial dependence is affected by rotation age choices. This is obtained by differentiating the functions in Definition 1 with respect to own-stand rotation ages. The resulting second derivatives define how spatial dependence between stands evolves with own rotation age. This is called 'temporal dependence' in the literature. That is,

Definition 2 (Koskela and Ollikainen 2001b). *Temporal Dependence*

$$F^a_{\tau T}(T,\tau)\begin{Bmatrix}<\\=\\>\end{Bmatrix}0\,;\; F^b_{T\tau}(T,\tau)\begin{Bmatrix}<\\=\\>\end{Bmatrix}0 \text{ if } \begin{cases}\text{stand dependence decreases with stand age}\\ \text{stand dependence unchanged with stand age}\\ \text{stand dependence increases with stand age}\end{cases}$$

From Definition 2, the temporal interdependence between two stands may be constant, increasing or decreasing depending on how spatial dependence between the stands changes with increases in the rotation age of each stand. *Temporal independence* results when $F^a_{\tau T}(T,\tau) = F^b_{T\tau}(T,\tau) = 0$. This is the case if spatial substitutability or complementarity (from Definition 1) is merely associated with site-specific properties which remain the same regardless of own-stand rotation age. *Increasing temporal dependence* between the stands means that, for spatial complements, the complementarity between stands increases with own-stand rotation age. But for spatial substitutes, the substitutability between stands decreases with own-stand rotation age. *Decreasing temporal dependence* implies just the opposite: complementarity weakens while substitutability becomes stronger for increases in own-stand rotation age.[3]

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[2] For the concept of the Auspitz-Liebig-Edgeworth-Pareto (ALEP) complementarity/ substitutability, see Samuelson (1974) and further discussions in Chipman (1977), Kannai (1980) and Weber (2000).

[3] Ecologists have shown that amenity production depends on inter-stand relationships which form the basis of Definitions 1 and 2. See, for example Franklin and Forman (1987) and Giles (1978).

2.1 Three models for rotation age

We now depart from the literature and consider rotation age solutions for different ownership structures and timing of decisions. The first rotation age solution comes from a Nash game, where there are two different landowners who own stands *a* and *b*, and these landowners make their harvesting decisions simultaneously taking the other's action as given. This mimics the private market solution where landowners are both price takers, and it requires no commitment to actions on the part of either landowner. The second rotation age solution follows when there are two landowners, but one is a first-mover, i.e., landowners play a traditional two-stage Stackelberg game. One landowner here is able to credibly commit to a harvesting decision before the other landowner moves, so that a leader-follower relationship is established. Finally, the third rotation age solution is derived under the assumption of a sole owner of both stands *a* and *b*.

In all cases, landowners are assumed to be price takers and, as such, do not account for price-induced demand changes when making rotation age choices. Each landowner does, however, utilize amenities produced by the other stand. Therefore, the sole owner model, by yielding the efficient solution, provides a hint at the social costs associated with uncoordinated harvesting.

2.2. Rotation ages in the Nash game

Here each landowner chooses rotation age taking the other landowner's rotation choice as given. The solution to the Nash game can be obtained for each landowner by solving the following simultaneous choice problem:

$$\underset{\{T\}}{Max}\,\Omega^{aN} = V^a + E^a \tag{3a}$$

$$\underset{\{\tau\}}{Max}\,\Omega^{bN} = V^b + E^b\ , \tag{3b}$$

where the terms in the objective functions are defined in (1) and (2). The labels *a* and *b* again refer both to the stand and landowner, and N denotes the Nash game.

The following first-order conditions characterize the optimal rotations T and τ:

$$\Omega_T^{aN} = 0: \qquad pf'(T) + F^a(T,\tau) = rpf(T) + rV^a + rE^a \tag{4a}$$

$$\Omega_\tau^{bN} = 0: \qquad qg'(\tau) + F^b(T,\tau) = rqg(\tau) + rV^b + rE^b\,. \tag{4b}$$

These suggest that both landowners equate their private marginal benefit of delaying harvest (LHS) to the marginal opportunity cost of delaying harvest (RHS). Notice there is an externality evident in the first-order conditions (affecting the last terms on the LHS and RHS of (4a) and (4b)). This arises because landowners do not account for the effect of their rotation age choice on the other landowner's utility and behavior.

The second-order conditions for both landowners are given by

$$\Omega_{TT}^{aN} = pf''(T) - rpf'(T) + F_T^a(T,\tau) < 0 \tag{5a}$$

$$\Omega_{\tau\tau}^{bN} = qg''(\tau) - rqg'(\tau) + F_\tau^b(T,\tau) < 0 \tag{5b}$$

We assume that the second-order conditions (5a) and (5b) hold. They hold automatically when the landowner's marginal amenity valuation decreases or remains constant with increasing own-stand rotation age, i.e., for $F_T^a \leq 0$ and $F_\tau^b \leq 0$ They may not always hold when the landowner's amenity valuation increases with own-stand rotation ages, i.e., for $F_T^a > 0$ and $F_\tau^b > 0$ (see Strang 1983).

As for the dynamics of the Nash equilibrium, we assume that landowners adjust their rotation ages in order to increase the sum of net present value of harvest revenue and amenities, taking the behavior of the other landowner as given. These dynamics are represented by the following derivatives,

$$\dot{\Omega}^{aN} = \lambda^T \left[\frac{\partial \Omega^{aT}}{\partial T}\right] \text{ and } \dot{\Omega}^{bN} = \lambda^\tau \left[\frac{\partial \Omega^{bT}}{\partial \tau}\right] \tag{5c}$$

where the parameters $\lambda^T, \lambda^\tau > 0$ indicate the speed of adjustment, and dots indicate derivatives of the target function with respect to (calendar) time.

The uniqueness and stability condition for the Nash game can be expressed as

$$\Delta^N = \Omega^{aN}_{TT}\Omega^{bN}_{\tau\tau} - \Omega^{aN}_{T\tau}\Omega^{bN}_{\tau T} > 0, \quad (5d)$$

where $\Omega^{aN}_{T\tau} = F^a_\tau(T,\tau) - r(1-e^{-rT})^{-1}\int_0^T F^a_\tau(s,\tau)e^{-rs}ds$

and $\Omega^{bN}_{\tau T} = F^b_T(T,\tau) - r(1-e^{-r\tau})^{-1}\int_0^\tau F^b_T(T,x)e^{-rx}dx$,

so that the determinant of the second-order derivatives matrix in (5d) must be positive.[4] The second order conditions (5a) – (5b) imply that the first part of (5d) is positive. Therefore, the uniqueness and stability of the Nash game depends on the product of cross-derivatives $\Omega^{aN}_{T\tau}\Omega^{bN}_{\tau T}$, which jointly with the second-order conditions define the slopes of the reaction functions for the landowners.

The reaction functions for landowner a and b respectively can be obtained from the first-order conditions by totally differentiating them with respect to the rotation age of the adjacent stands,

$$\left.\frac{dT}{d\tau}\right|_a \equiv a(\tau) = -\frac{\Omega^{aN}_{T\tau}}{\Omega^{aN}_{TT}}, \qquad \left.\frac{d\tau}{dT}\right|_b \equiv b(T) = -\frac{\Omega^{bN}_{\tau T}}{\Omega^{bN}_{\tau\tau}}. \quad (6)$$

Lemma 1 characterizes how the cross derivatives of the reaction functions depend on properties of the amenity valuation functions.

Lemma 1. $\Omega^{aN}_{T\tau}\begin{Bmatrix}>\\=\\<\end{Bmatrix}0 \ \ if \ \ F^a_{\tau T}\begin{Bmatrix}>\\=\\<\end{Bmatrix}0 \ \ and \ \ \Omega^{bN}_{\tau T}\begin{Bmatrix}>\\=\\<\end{Bmatrix}0 \ \ if \ \ F^b_{T\tau}\begin{Bmatrix}>\\=\\<\end{Bmatrix}0.$

Proof. See Appendix 1.

According to Lemma 1 and equation (6), the reaction functions have different slopes depending on temporal stand interdependence (Definition 2). There are three cases, which we summarize in:

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4 Dixit (1986) and Vives (1999, pp. 49-58) discuss further details of the uniqueness and stability analysis we use here.

Result 1. Properties of the reaction functions

a) *Under temporally independent stands, the reaction functions are vertical lines in (T,τ) space and the equilibrium is stable.*
b) *Under increasing temporal dependence, the reaction functions are increasing in (T,τ) space. Stability of the equilibrium requires that the reaction function for stand a is steeper than the reaction function for stand b.*
c) *Under decreasing temporal dependence, the reaction functions are decreasing in (T,τ) space. Stability of the equilibrium requires that the reaction function for stand a is steeper than the reaction function for stand b.*

Result 1 is illustrated in Figures 1-4 for both decreasing and increasing temporal dependence. Drawn in the figures are the reaction functions for the landowner of stand a and b, $a(\tau)$ and $b(T)$, each of which is a function of the rotation age choice of the other landowner. In Figures 1–2 the reaction functions are downward-sloping, reflecting decreasing temporal dependence between the stands. The upward sloping reaction functions in Figures 3 and 4 reflect increasing temporal dependence between the stands. As we explain later, Figures 1 and 3 are drawn assuming the stands are spatial substitutes, while Figures 2 and 4 are drawn assuming the stands are spatial complements. The unique and stable Nash equilibrium solution satisfying (4a) and (4b) occurs at the point where the reaction functions for the landowners cross.

2.3. Rotation ages in the Stackelberg game

In the Stackelberg game, the leader moves with knowledge of how the following-landowner responds; the follower takes the rotation age of the leader as given. While this model may mimic some private market situations, the leader might be interpreted as a government formally setting a long run harvest policy, effectively leading.

Assume the leader is the landowner holding stand *a*. This landowner maximizes the following objective function,

$$\underset{\{T\}}{Max}\ \Omega^{aS} = V^a + E^a \tag{7a}$$

$$s.t. \quad \tau^S = \tau(T^S, q, r, c_\tau), \tag{7b}$$

Figure 1. Decreasing temporal dependence when the stands are spatial substitutes

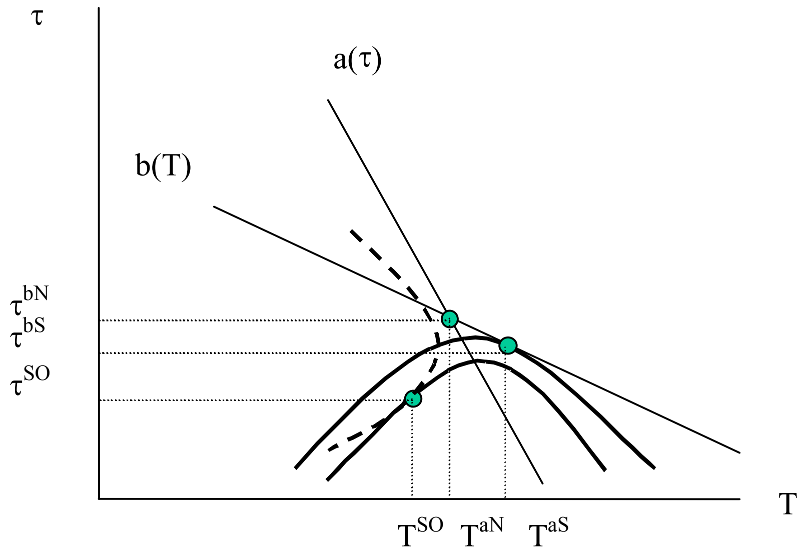


Figure 2. Decreasing temporal dependence when the stands are spatial complements

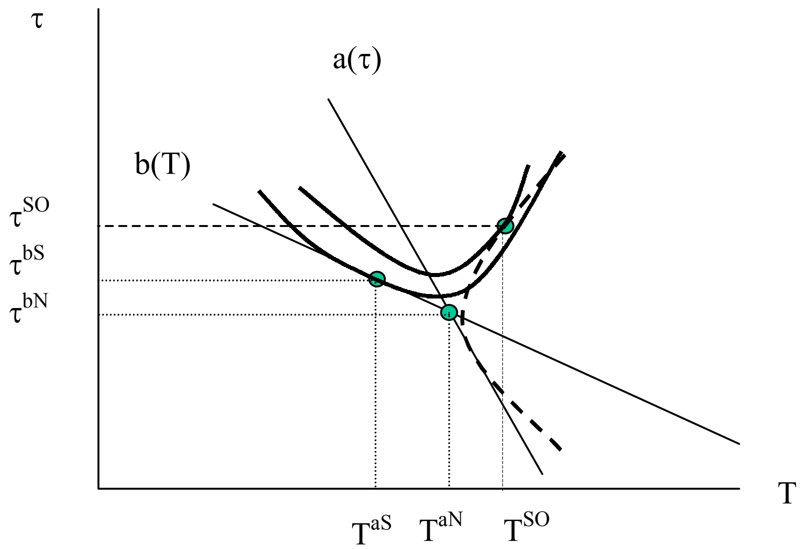


Figure 3. Increasing temporal dependence when the stands are spatial substitutes

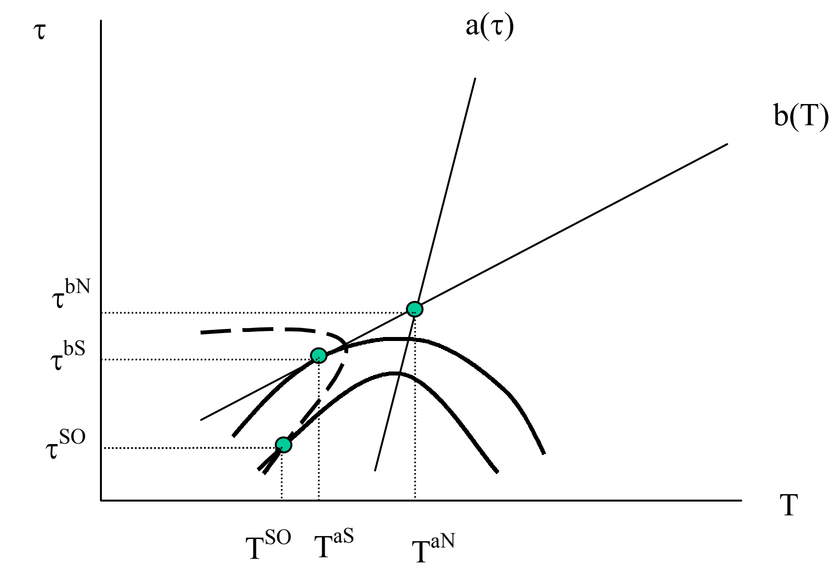
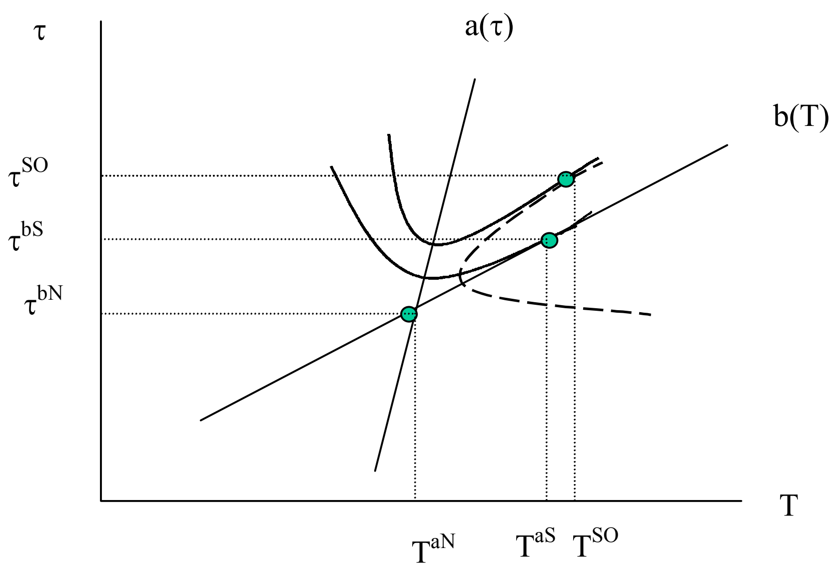


Figure 4. Increasing temporal dependence when the stands are spatial complements



where the superscript S refers to the Stackelberg game, and $\tau^S = \tau(T^S, q, r, c_\tau)$ describes the reaction function of the follower (who holds stand *b*). Utilizing the Nash first-order condition (3a) for the follower's reaction function, the leader and the follower first-order conditions are, respectively,

$$\Omega_T^{aS} = \Omega_T^{aN} + e^{rT}\tau_T(.)\int_0^T F_\tau^a(s,\tau)e^{-rs}ds = 0, \tag{8a}$$

$$\Omega_\tau^{bS} = qg'(\tau) + F^b(T,\tau) - rqg(\tau) - rV^b - rE^b = 0. \tag{8b}$$

We assume that the second-order condition holds for this problem. In reality, whether it holds depends again on the amenity valuation function and on the properties of the follower's reaction function $\tau^S = \tau(T^S, q, r, c_\tau)$.[5]

Consider first the *follower's* behavior given in (8b). This condition is qualitatively the same as the necessary condition for landowner *b* in the Nash game; that is, given T^S, the follower chooses the rotation age τ. This is not so for the leader (eqn (8a)). Compared to the Nash game, there is an additional term in the Stackelberg first order condition, reflecting the impact of the follower's rotation choice on the leader's marginal amenity benefits. The presence of this additional term implies that the leader partly accounts for the externality that arises from the effect of the follower's rotation age on the amenities of the leader's stand. This interesting difference between Stackelberg and Nash outcomes will become important later when we study comparative statics effects.

Whether the leader has a longer or shorter rotation age compared to the Nash equilibrium rotation age depends on the last term in (8a), i.e., on the slope of the follower's reaction curve and the integral term. To sign this integral term and make the analysis tractable, we assume that the amenity function has a quadratic shape, and

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[5] This can be seen from

$$\Omega_{TT}^{aS} = \Omega_{TT}^{aN}e^{-rT} - r(pf'(T) - rpf(T))e^{-rT} + F^a(T,\tau)$$

$$+\tau_{TT}\int_0^T F_\tau^a(s,\tau)e^{-rs}ds + \tau_T^2\int_0^T F_{\tau\tau}^a(s,\tau)e^{-rs}ds + 2\tau_T F_\tau^a(T,\tau)e^{-rT} < 0$$

we use a second order approximation for $e^{-rT} = 1/(1 + rT + (1/2)r^2T^2)$.[6] The quadratic shape happens to be convenient, because it allows for all relevant cases described in Definitions 1 and 2.

Lemma 2. *Under the quadratic amenity valuation function,*

$F^a(T,\tau) = \alpha(T+\tau) - \frac{1}{2}\beta(T+\gamma\tau)^2$, *we have*

$$\int_0^T F_\tau^a(s,\tau)e^{-rs}ds \begin{Bmatrix} > \\ < \end{Bmatrix} 0 \text{ as } F_\tau^a(0,\tau) \begin{Bmatrix} > \\ < \end{Bmatrix} 0.$$

Proof. See Appendix 2.

According to Lemma 2 the sign of the integral term depends how the rotation age of the follower's stand affects the marginal amenity valuation of the leader's stand at the margin, when $T = 0$. Returning to Figures 1-4, consider now the iso-net-present-value-of-revenue curves. Iso-net-present-value-of-revenue curves are lines along which net present value is constant for given interest rates, timber prices and regeneration costs. A family of these therefore exist for each set of constant parameters.

When the stands are spatial substitutes, the iso-net-present-value-of-revenue curves are decreasing in the rotation age of the other stand, while spatial complements implies the iso-net-present-value-of-revenue curves are increasing in the rotation age of the other stand. This means that, for complements (substitutes) the net present value of profits for the forest landowner is increasing when moving up (down) the reaction functions. In the figures, the Stackelberg rotation age for leader and follower corresponding to (8a) – (8b) above is defined by the point where the leader's highest iso-net-present-value-of-revenue curve is tangent to the follower´s reaction function, b(T).

Using Figures 1-4, Lemma 2, Lemma 1, and Result 1, we can now examine the relationship between the Nash and the Stackelberg rotation ages. Consider first the leader. If the stands are temporally independent, then $F_{\tau T}^a(.) = 0$ and the reaction functions are vertical lines. Here, the sign of $F_\tau^a(T,\tau)$ does not matter, therefore, the Stackelberg rotation age coincides with the Nash rotation age. Under decreasing

[6] We can show that higher order approximations for the discount factor will not change the nature of the results in Lemma 2 below.

(increasing) temporal dependence and spatial substitutability between the stands, we have $\Omega_T^{aN} < 0$ ($\Omega_T^{aN} > 0$), because now the last term in (8a) is positive (negative). In this case, the leader's rotation age, T^S, is longer (shorter) than the Nash rotation age, T^N. If the stands are spatial complements, then under decreasing (increasing) temporal dependence we have $\Omega_T^{aN} > 0$ ($\Omega_T^{aN} < 0$); now the leader's rotation age is shorter (longer) than the Nash rotation age.

For spatial substitutes (complements), the follower's rotation age is shorter (longer) than the Nash rotation age under decreasing temporal dependence. The interpretation is as follows. The follower observes the rotation age of the leader prior to moving. When the stands are substitutes, the follower's rotation age must be shorter because, under decreasing temporal dependence, the leader's rotation age will be longer than the Nash rotation age, and the follower's reaction curve will be downward-sloping (Figure 1). Intuitively, the longer rotation age of the leader allows the follower to harvest sooner but still derive foregone amenity benefits from the leader's stand. For spatial complements, the leader's rotation age is shorter than the Nash age, and the best response of the follower is to lengthen the rotation age relative to Nash age, because that decreases stand complementarity. Similar reasoning can be applied to increasing temporal dependence.

We can summarize the above discussion in:

Proposition 1. *The relationship between Nash and Stackelberg rotation ages depends on the nature of stand interdependence:*

a) Under temporal independence, $T^N = T^S$ *and* $\tau^N = \tau^S$.

b) Under decreasing temporal dependence, $T^N > T^S$ *and* $\tau^N < \tau^S$ *for spatial complements, while* $T^N < T^S$ *and* $\tau^N > \tau^S$ *for spatial substitutes.*

c) *Under increasing temporal dependence,* $T^N < T^S$ *and* $\tau^N < \tau^S$ *for spatial complements, while* $T^N > T^S$ *and* $\tau^N > \tau^S$ *for spatial substitutes.*

2.4. Rotation ages for the sole owner

The sole owner chooses rotation ages of both stands to maximize joint economic rents,

$$\underset{\{T,\tau\}}{Max}\, W = V^a + V^b + E^a + E^b \ . \tag{9}$$

The first-order conditions characterizing sole owner rotation age choices can be expressed using a modification of the Nash conditions,

$$W_T = \frac{e^{-rT}}{(1-e^{-rT})}\Omega_T^{aN} + \frac{\int_0^{\tau} F_T^b(T,x)e^{-rx}dx}{(1-e^{-r\tau})} = 0 \tag{10a}$$

$$W_\tau = \frac{e^{-r\tau}}{(1-e^{-r\tau})}\Omega_\tau^{bN} + \frac{\int_0^{T} F_\tau^a(s,\tau)e^{-rs}ds}{(1-e^{-rT})} = 0 \tag{10b}$$

The second-order conditions $W_{TT} < 0$, $W_{\tau\tau} < 0$, and $\Delta^{SO} > 0$ are presented in Appendix 3 and are assumed to hold.

Equations (10a) and (10b) imply that the sole owner chooses rotation ages for both stands taking into account how amenities are affected by rotation age, not only of stand *a* (as in the Stackelberg game) but also of stand *b* (see the last term in 10a and 10b). Hence, all potential externalities arising from the effects of harvesting one stand on the other stand's amenities are internalised. The sole owner outcome is therefore the efficient solution for our problem.

The last terms in equations (10a) and (10b) determine how the sole-owner rotation age of both stands compares to Nash and Stackelberg rotation ages. From Lemma 2 we know that these last terms are positive when the stands are spatial complements and negative when the stands are spatial substitutes. Thus, relative to the Nash rotation age, the sole owner chooses longer rotation ages for both stands when they are spatial complements, but shorter rotation ages when they are spatial substitutes.

How does the sole owner rotation age compare to the Stackelberg rotation age? The sole owner first order conditions differ from the Stackelberg conditions by the last term (compare (10a) with (8a)). Due to the symmetric sole owner first-order conditions (10a) and (10b), we can graphically distinguish the sole owner optimum in Figures 1-4 as points where the iso-net-present-value-of-revenue-curves from both stands are *tangent* to each other. Referring to the figures, for spatial complements (substitutes) the sole owner rotation age is longer (shorter) than the Stackelberg age for both leader and follower.

The important driving factor in the comparison of the sole owner's rotation age with Nash and Stackelberg rotation ages is the spatial complementarity or substitutability of stands. The sole owner internalizes all externalities associated with amenities. When stands are temporally independent, it is natural that the sole owner rotation age coincides with the other rotation ages because there is no external effect of harvesting one stand on the other stand. However this is not the case when stands are temporally interdependent. Now the comparison of rotation ages depends on how the two stands are related spatially; if they are spatial complements, the sole owner increases rotation ages, while the opposite is true under spatial substitutes. We can express the relationship between Nash, Stackelberg and sole owner rotation ages as follows

Proposition 2.

a) Under temporal independence, the spatial complementarity or substitutability between the stands does not matter and Nash and Stackelberg rotation ages coincide with the sole owner rotation ages.

b) Under temporal dependence, the sole owner rotation age is longer (shorter) than Nash and Stackelberg solutions when stands are spatial complements (substitutes).

The differences in rotation ages under our various solutions will undoubtedly lead to differences in welfare for forest landowners. Obviously, landowners are by definition better off at the sole owner optimum relative to the other outcomes. An interesting comparison of welfare under the other outcomes can be obtained from Figures 1-4, by noting the position of the equilibria on the iso-net-present-value-of-revenue curves. Figures 3-4 show that when temporal dependence is increasing, the welfare of both landowners is higher under the Stackelberg solution than under the Nash equilibrium. This result occurs because in the Stackelberg game, the leader partially accounts for the effects of his rotation age decision on the follower landowner, and this increases the welfare of both leader and follower. In the case of decreasing temporal dependence, Figures 1-2 show that the welfare comparison between Nash and Stackelberg solutions is ambiguous. Thus, we have an additional Corollary:[7]

Corollary 1. *Under increasing temporal independence, both landowners are better off when rotation ages are chosen according to the Stackelberg game, relative to the case when rotation ages are solved under a Nash equilibrium.*

3. Comparative Static Analysis

We have shown that the Nash, Stackelberg, and sole owner rotation ages differ from each other in the presence of amenity benefits. Now we study the qualitative properties of these rotation ages. An important point to realize is that market parameters for both stands can differ given that site characteristics inherent to both stands could differ, and their rotation ages are not generally equivalent, as we showed above. The results of this section are condensed in Table 1.

3.1 Nash game

For the effect of a timber price p of stand a on both rotation ages we obtain, through total differentiation of the first order conditions,

$$T_p^N = -\Delta^{N^{-1}} \left\{ \Omega_{Tp}^{aN} \Omega_{\tau\tau}^{bN} \right\} \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0 \text{ as } A \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0, \tag{12a}$$

$$\tau_p^N = \Delta^{N^{-1}} \left\{ \Omega_{Tp}^{aN} \Omega_{T\tau}^{bN} \right\}, \tag{12b}$$

[7] Hamilton and Slutsky (1990) have shown, in models of endogenous timing regarding the decisions of firms, there are incentives for firms to move sequentially if there is one Stackelberg equilibria that Pareto dominates all other simultaneous move Nash equilibria. With two firms, the necessary condition for the Stackelberg to obtain is that the leader's profits are higher when moving first, compared to it profits in the Nash game, otherwise no firm would choose to move first and both would, effectively, play a Nash game. A sufficient condition for this is that both leader and follower profits in the sequential move game is at least high as their Nash profits.

where $\Omega_{Tp}^{aN} = f'(T) - rf(T) - rf(T)e^{-rT}(1-e^{-rT})$, $A = rc(1-e^{-rT})^{-1} + F^a(T,\tau) - rE^a$

and $F^a(T,\tau) - rE^a \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0$ as $F_T^a(T,\tau) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0$ (see Koskela-Ollikainen 2001b).

Given that the present value of regeneration costs is always positive, we have the conventional effect of a shorter rotation age due to an increase in the own-stand harvesting price when $F_T^a(T,\tau) \geq 0$ (i.e., the marginal amenity valuation increases or remains constant with the age of trees).

Interestingly, (12b) reveals that the rotation age of stand *b* may also be affected by a change in stand *a*'s price. Only if the stands are temporally independent, i.e., $\Omega_{T\tau}^{bN} = 0$, will the landowner of stand *b* not change his rotation age if the price of stand *a* changes. Assuming that the marginal amenity valuation does not decrease with the age of the own stand, the other landowner will shorten (lengthen) his rotation age as a result of a rise in p when temporal dependence between stands is increasing (decreasing). Note that the signs of T_q^N and τ_q^N are symmetric given (12a) and (12b).

To assess the impacts of a change in regeneration costs c_T on rotation ages, we obtain

$$T_{c_T}^N = -\Delta^{N^{-1}} \left\{ \frac{r}{(1-e^{-rT})} \Omega_{TT}^{aN} \right\} > 0 \tag{13a}$$

$$\tau_{c_T}^N = \Delta^{N^{-1}} \left\{ \frac{r}{(1-e^{-rT})} \Omega_{\tau T}^{bN} \right\} \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \text{ as } \Omega_{\tau T}^{bN} \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0. \tag{13b}$$

Thus, the effect of an increase in the own-stand regeneration cost is qualitatively the same as in the Faustmann and Hartman model-based literature. However, with respect to the adjacent stand, the Nash solution brings a new result, from (13b). The reaction of the adjacent stand's landowner to a change in the regeneration costs of the other landowner depends on the temporal dependence between stands. More specifically, if the dependence between the stands increases (decreases) with a longer rotation age for the own stand, then the owner of the adjacent stand lengthens (shortens) his rotation age. Again, the signs of $T_{c_\tau}^N$ and $\tau_{c_\tau}^N$ are symmetric given (13a) and (13b).

Finally, for a change in the real interest rate, we totally differentiate the Nash first order conditions to obtain,

$$T_r^N = -\Delta^{N^{-1}}\left\{\Omega_{Tr}^{aN}\Omega_{\tau\tau}^{bN} - \Omega_{\tau r}^{bN}\Omega_{T\tau}^{aN}\right\} \quad (14a)$$

$$\tau_r^N = -\Delta^{N^{-1}}\left\{\Omega_{\tau r}^{bN}\Omega_{TT}^{aN} - \Omega_{Tr}^{aN}\Omega_{\tau T}^{bN}\right\}, \quad (14b)$$

where $\Omega_{Tr}^{aN} < 0$ and $\Omega_{\tau r}^{bN} < 0$. Under increasing temporal dependence both rotation ages will unambiguously shorten from (14a) and (14b). However, under decreasing temporal dependence the effect is a priori ambiguous. Naturally, a sufficient condition for a shorter rotation age here is that the own-stand direct effect of the interest rate dominates all other effects. Summarizing we have

Result 2. *In a symmetric Nash equilibrium with interdependent stands,*

a) A higher own-stand price shortens the rotation age under increasing temporal dependence, but may increase rotation age under strong decreasing temporal dependence. The effect of own-stand regeneration cost on rotation age is positive.

b) The effects of higher adjacent-stand timber price and regeneration costs on the own-stand rotation age depend on the nature of temporal dependence between the stands.

c) *A higher interest rate decreases rotation ages under increasing temporal dependence, but is a priori ambiguous under decreasing temporal dependence*

3.2 Stackelberg game

In the Stackelberg game we have only to solve the comparative statics for the leader's rotation age.[8] A change in the timber price for stand *a* impacts the leader's rotation age as follows

[8] Comparative statics for the follower has been analyzed in Koskela-Ollikainen (2001b).

$$T_p^S = -\frac{\Omega_{Tp}^{aS}}{\Omega_{TT}^{aS}} \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0 \; as \; A \quad + \tau_T(.) \int_0^T F_\tau^a(s,\tau) e^{-rs} ds \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0, \qquad (15)$$

where $\Omega_{Tp}^{aS} = f'(T) - rf(T) - rf(T)e^{-rT}(1-e^{-rT})$ and A is defined in (12a).

Recall from Result 1 that the sign of the reaction function $\tau_T(.)$ depends on temporal dependence between the stands, and that from Lemma 2 the sign of the integral term depends on whether the stands are spatial complements or substitutes. We can first see that if the stands are temporally independent ($F_{T\tau}^a = 0$), then the last term in (15) is zero and the price effect on the Stackelberg rotation age is identical to the price effect on rotation age in the single stand Hartman model. Second, a combination of increasing marginal valuation, spatial complements and increasing temporal dependence, or spatial substitutes with decreasing temporal dependence, implies unambiguously the conventional effect of a shorter rotation age.

Given that the Stackelberg game is not symmetric like the Nash game**,** we next solve for the effect of the price of stand *b* on the leader's rotation age. Differentiating the first-order condition (8a) with respect to this price (q) and noting that all effects emerge from the follower's response function, we obtain

$$T_q^S = -\frac{\Omega_{Tq}^{aS}}{\Omega_{TT}^{aS}}. \qquad (16)$$

We show in Appendix 4 that, while this price effect is a priori ambiguous for both decreasing and increasing temporal dependence, it is zero for temporally independent stands. The ambiguity under non-constant temporal dependence results from the fact that the direct and indirect effects of timber price q are offsetting (via a shift in the follower's reaction function and its slope).

The effect of higher regeneration cost of the leader's stand on the leader's rotation age is given by

$$T_{c_T}^S = -\frac{r}{(1-e^{-rT})\Omega_{TT}^{aS}} > 0. \qquad (17a)$$

$$T^S_{c_\tau} = 0\,, \tag{17b}$$

Hence, the leader's rotation age unambiguously lengthens. The interpretation of this finding is the same as before. If the regeneration cost of the follower's stand increases, it affects the leader's first-order condition only via the reaction function of the follower. Given that $\tau_{Tc_\tau}(.) = 0$, there is no effect on the rotation age of the leader's stand.

Finally, turning to the effect of a higher interest rate on the leader's rotation age, we can show that

$$T^S_r = -\frac{\Omega^{aS}_{Tr}}{\Omega^{aS}_{TT}}\,, \tag{18}$$

where

$$\Omega^{aS}_{Tr} = -\left(1 + \frac{Te^{-rT}}{(1-e^{-rT})}\right)\left[pf(T) + V^a + E^a\right] - \int_0^T sF(s,\tau)e^{-rs}ds$$
$$-\tau_T \int_0^T sF_\tau(s,\tau)e^{-rs}ds + \tau_T \tau_r \int_0^T F_{\tau\tau}(s,\tau)e^{-rs}ds + \tau_{Tr} \int_0^T F_\tau(s,\tau)e^{-rs}ds$$

According to this expression for Ω^{aS}_{Tr}, the interest rate effect arises through three channels: directly i) through the profitability of the leader's stand, and indirectly both ii) via the slope of the reaction function, and iii) through the position of the follower's reaction function. In Appendix 3 we show that these effects counter each other, so that the overall impact of the interest rate is ambiguous.

We can now summarize our findings in

Result 3. *In a Stackelberg equilibrium with interdependent stands,*

a) *Under increasing marginal amenity valuation, an increase in the leader's own-stand price will shorten the leader's rotation age when the stands are spatial complements with increasing temporal dependence, or spatial substitutes with decreasing temporal dependence. A higher own-stand regeneration cost increases the leader's rotation age unambiguously.*
b) *An increase in the follower's own-stand price has an a priori ambiguous effect on rotation age when stands are not temporally independent, while higher own-stand regeneration costs for the follower have no effect on the leaders' rotation age.*

c) *A higher interest rate has an a priori ambiguous effect on the leader's rotation age.*

3.3. Sole owner solution

Finally, we move on to the comparative static effects in the sole owner case. For the effect of higher timber price of stand *a* on both rotation ages we have

$$T_p^{SO} = -\Delta^{SO^{-1}}\{W_{Tp}W_{\tau\tau}\} \tag{19a}$$

$$\tau_p^{SO} = \Delta^{SO^{-1}}\{W_{Tp}W_{\tau T}\}\ , \tag{19b}$$

where $W_{Tp} = \dfrac{e^{-rT}}{(1-e^{-rT})}\left[f'(T) - rf(T) - \dfrac{rf(T)e^{-rT}}{(1-e^{-rT})}\right]$, and

$$W_{\tau T} = \frac{e^{-r\tau}}{(1-e^{-r\tau})}\Omega_{T\tau}^{aN} + \frac{e^{-rT}}{(1-e^{-rT})}\left[F_\tau^a(T,\tau) - \frac{r\int_0^T F_\tau^a(s,\tau)e^{-rs}ds}{(1-e^{-rT})}\right].$$

We can show that $\operatorname{sgn} W_{Tp}$ = -sgn$\left[A + \int_0^\tau F_T^b(T,x)e^{-rx}dx\right]$, where $A = rc(1-e^{-rT}) + F^a(T,\tau) - rE^a$. The sufficient conditions for $W_{Tp} < 0$, and the impact of own-stand price on the rotation age to be negative, are that marginal amenity valuations are non-decreasing with the rotation age of the stand $(F_T^a(.) \geq 0)$ and the stands are spatial complements $(F_T^b(.) > 0)$ (see Lemma 2). Under the sufficient condition for $W_{Tp} < 0$, the effect of the adjacent stand price on own-stand rotation age is negative under increasing temporal dependence and positive under decreasing temporal dependence; because according to Lemma 1 $W_{\tau T} >,< 0$, as the temporal dependence between stands increases, or decreases, respectively. Thus, the price effects resemble the classic case of complements and substitutes, except in our model complementarity is specified in a temporal sense.

The effect of a change in the own-stand regeneration costs of stand *a* on the rotation age is given by,

$$T_{c_T}^{SO} = -\Delta^{SO^{-1}} \left\{ \frac{re^{-rT}}{(1-e^{-rT})^2} W_{TT} \right\} > 0 \tag{20a}$$

$$\tau_{c_T}^{SO} = \Delta^{SO^{-1}} \left\{ \frac{re^{-rT}}{(1-e^{-rT})^2} W_{T\tau} \right\} \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} > 0 \text{ as } W_{T\tau} \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 . \tag{20b}$$

A higher own-stand regeneration cost lengthens its rotation age, while the effect on the other stand depends again on the temporal dependence between stands. The rotation age of the other stand lengthens under increasing temporal dependence and shortens under decreasing temporal dependence.

Finally, for the effects of an increase in the real interest rate we have

$$T_r^{SO} = -\Delta^{SO^{-1}} \{ W_{Tr} W_{\tau\tau} - W_{\tau r} W_{T\tau} \} \tag{21a}$$

$$\tau_r^{SO} = -\Delta^{SO^{-1}} \{ W_{\tau r} W_{TT} - W_{Tr} W_{\tau T} \}, \tag{21b}$$

where

$$W_{Tr} = \frac{e^{-rT}}{(1-e^{-rT})} \Omega_{Tr}^{aN} + \frac{T}{(1-e^{-rT})^2} \left\{ (1-e^{-rT}) \int_0^\tau F_T^b(T,x) e^{-rx} dx - \frac{1}{T} \int_0^\tau x F_T^b(T,x) e^{-rx} dx \right\}$$

$$W_{\tau r} = \frac{e^{-r\tau}}{(1-e^{-r\tau})} \Omega_{\tau r}^{bN} + \frac{\tau}{(1-e^{-r\tau})^2} \left\{ (1-e^{-rT}) \int_0^T F_\tau^a(s,\tau) e^{-rs} ds - \frac{1}{\tau} \int_0^T s F_\tau^a(s,\tau) e^{-rs} ds \right\}$$

We know that the first terms in W_{Tr} and $W_{\tau r}$ are negative. In Appendix 5 we show that, under plausible assumptions concerning the interest rate and rotation ages the braced terms are negative if the stands are spatial complements. Thus**,** under increasing temporal dependence and spatial complementarity of stands, both rotation ages will unambiguously shorten; otherwise the effects are a priori ambiguous. We therefore have,

Result 4. *In the case of a sole owner with interdependent stands,*

a) A higher own-stand price shortens rotation age if stands are either spatial complements, and temporal dependence is either unchanged or increasing. A higher own-stand regeneration cost unambiguously lengthens the rotation age

b) Under a certain sufficient condition, the effect of the adjacent stand price on own-stand rotation age depends on the nature of temporal dependence. The effect of *the adjacent stand's regeneration cost will increase (decrease) the own-stand rotation age under increasing (decreasing) temporal dependence.*

c) *If stands are spatial complements and temporal interdependence is increasing, then a higher interest rate will shorten the rotation ages.*

4. Discussion and Policy Implications

Sustaining forest ecosystems requires that stands are managed in concert rather than in isolation. Most forest economics models, however, consider only a single isolated stand or a single landowner. This is not necessarily the case for private land ownership, where individual property rights make the proper coordination of management decisions across large numbers of landowners very difficult. The lack of coordination among landowners can be detrimental to amenities that depend on the ecosystem as a whole, such as those derived from recreational experiences or the existence of certain wildlife species.

We examine the possibility that stands can be temporally or spatially interdependent in different ways regarding the production of amenities. We then allow for several cases of landowner decision timing and commitment, including landowners making simultaneous decisions or one acting as a first mover. Our results extend the basic Hartmann model of forest management that first introduced amenities for the case of an isolated stand, as well as other models of multiple stands based on numerical simulations or the assumption of only sole ownership.

We demonstrate that rotation age decisions depend on how adjacent stands are spatially and temporally dependent with regards to amenity valuation, and on the structure of landowner decision timing. Sole ownership represents the social optimum in our model. Comparing this with the Nash and Stackelberg outcomes gives a qualitative indication of the social costs associated with landowners who do not seek to jointly maximize total revenue and amenity rents from owning forests.

The collective results from our paper are summarized in Propositions 1, 2, Corollary 1, and Table 1. The sole owner's rotation age is longer than the Nash and Stackelberg rotation ages if the stands are spatial complements, but shorter if they are spatial substitutes with regard to marginal amenity valuation. Additionally, the relationship between the Nash and Stackelberg rotation ages also depends on how this substitutability and complementarity evolves over time. The differences between these solutions largely reflect the ability (or inability) of landowners to either benefit from another landowner's decisions. Interestingly, we show that under increasing temporal interdependence, it can be the case that one landowner moving first may make both landowners better off relative to the case where they do not coordinate at all and play a Nash game.

As a basis for policy analysis, we also characterize in these new models how rotation ages depend on timber prices, interest rates, and regeneration costs. We find that the conventional wisdom regarding effects of these parameters, derived in single stand models, does not usually hold. Instead, the results depend on the spatial and temporal dependence between the stands, the ability of landowners to commit to harvesting, and whether the parameters changing are for the stand in question or the adjacent stand. By and large, most of the differences between our models and existing models occur for two reasons. First, the possibility of increasing or decreasing temporal dependence often determines the signs of comparative statics results, for instance, the interest rate effect on rotation age (for instance) is not necessarily negative like in existing models with one forest stand (see Table 1). Second, stand interdependence implies that parameters from one stand can affect the choices made in the other stand, as exemplified by our price and regeneration cost effects.

Our new results and approach to studying landowner behavior suggests that existing models of policy design should be revised. We demonstrate that both Nash and Stackelberg equilibrium rotation ages are longer than sole owner rotation ages when stands are spatial substitutes, but Nash and Stackelberg ages are shorter than sole owner ages when stands are spatial complements. Clearly the scope for using taxes or subsidies to adjust rotation ages toward their efficient levels depends on the nature of stand interdependence regarding amenities, and also on the ability of landowners to commit to rotation age actions. These ideas have not been previously uncovered, even within the spatial forestry literature. In the end, the design of a proper Pigouvian tax system which mimics the efficiency of the sole owner solution will be much more complicated than previously thought. Results obtained depend on critically on the nature of interdependence between stands. What we can learn from empirical work regarding this interdependence will be crucial to practical policy work.

Literature Cited

Albers, H. 1996. Modeling Ecological Constraints on Tropical Forest Management: Spatial Interdependence, Irreversibility and Uncertainty. *Journal of Environmental Economics and Management* 30, 73-94.

Ando A, J. Camm, S. Polasky and A. Solow 1998. Species Distribution, Land Values and Efficient Conservation. *Science* 279, 2126-2128.

Beavers M., J. Hopf, B. Kent and M. Raphael 1995. Sustainable Forest Management for Optimizing Multispecies Wildlife Habitat: A Coastal Douglas-Fir Example. *Natural Resource Modelling* 1, 1-24.

Beavers M. and J. Hopf 1999. Spatially Optimizing Wildlife Habitat Edge Effects in Forest Management Linear and Mixed-Integer Programs. *Forest Science* 45, 249-258.

Bowes, M.and J. Krutilla. 1989. *Multiple-Use Management: The Economics of Public Forests.* Washington D.C., Resources for the Future. 353 p.

Bowes, M.and J.Krutilla. 1985. Multiple-Use Management of Public Forestlands. P. 531- 569 *in Handbook of Natural Resource and Energy Economics* vol. II, Kneese, A. and Sweeney, J. (eds.). North-Holland, Amsterdam.

Chipman, J. 1977. An Empirical Implication of Auspitz-Lieben-Edgeworth-Pareto Complementarity. *Journal of Economic Theory* 14, 228-231.

Csuti B., S. Polasky, P. Williams, R. Pressey, J: Camm, M. Kershaw, R. Kiester, B. Hamilton, M. Huso and K: Sahr (1997) A Comparison of Reserve Selection Algorithms Using Data on Terrestrial Vertebrates in Oregon. *Biological Conservation* 80, 83-97.

Dixit, A. 1986. Comparative Statics for Oligopoly. *International Economic Review* 27, 107-122.

Franklin, J., and R. Forman 1987. Creating landscape patterns by forest cutting: Ecological consequences and principles. *Landscape Ecology* 1,5-18.

Giles, R. 1978. *Wildlife Management*. Freeman Press, San Fransisco.

Hamilton, J., and S. Slutsky 1990. Endogenous Timing in Duopoly Games. *Games and Economic Behavior*, 2, 29-46.

Haight R., L. Travis 1997. Wildlife Conservation Planning Using Stochastic Optimization and Importance Sampling. *Forest Science*, 43, 129-139.

Hartman, R. 1976. The Harvesting Decision When a Standing Forest Has Value.

Economic Inquiry 14, 52-58.

Hopf, J. and M. Raphael 1993. Some Mathematical Programming Approaches for Optimizing Timber Age-Classes Distributions to Meet Multispecies Wildlife Population Objectives. *Canadian Journal of Forest Research* 23, 828-834.

Kannai, Y.1980. The ALEP Definition of Complementarity and Least Concave Utility Functions. *Journal of Economic Theory* 22, 115-117.

Koskela, E. and M. Ollikainen. 2001a. Forest Taxation and Rotation Age under Private Amenity Valuation: New Results. *Journal of Environmental Economics and Management* 42, 374-384.

Koskela, E. and M. Ollikainen. 2001b. Optimal private and Public Harvesting under Spatial and Temporal Interdependence. *Forest Science* 47, 484-496.

Ljunqvist, L. and T. Sargent 2000. *Recursive Macroeconomic Theory*. MIT Press. Cambridge, Mass.

Polasky, S., J. Camm and B. Garber-Yonts 2001. Selecting Biological Reserves Effectively: An Application to Terrestrial Vertebrate Conservation in Oregon. *Land Economics* 77, 68-78.

Samuelson, P. 1974. Complementarity. An Essay on the 40th Anniversary of the Hicks-Allen Revolution in Demand Theory. *Journal of Economic Literature* 12, 1255-1289.

Strang, W. 1983. On the Optimal Forest Harvesting Decision. *Economic Inquiry* 21, 576-583.

Swallow, S., P. Talukdar, and D.Wear. 1997. Spatial and Temporal Specialization in Forest Ecosystem Management under Sole Ownership. *American Journal of Agricultural Economics* 79, 311-326.

Swallow, S. and D. Wear. 1993. Spatial Interactions in Multiple-Use Forestry and Substitution and Wealth Effects for the Single Stand. *Journal of Environmental Economics and Management* 25, 103-120.

Weber, C.E. 2000. Two Further Empirical Implications of Auspitz-Lieben-Edgeworth-Pareto Complementarity. *Economics Letters* 67, 289-295.

Vives, X. 1999. *Oligopoly Pricing: Old Ideas and New Tools,* MIT Press. Cambridge, Mass.

TABLE 1. Comparative statics of the rotation age in Nash, Stackelberg, and sole owner Solutions for stand a.* A '+' indicates an increase in rotation age, while '-' indicates a decrease in rotation age. Marginal amenity function signs correspond to Definitions 1 and 2 in the text.

Exogenous Parameter	*Nash Rotation Age* (T^N)	*Stackelberg Rotation Age Leader* (T^S)	*Sole Owner Rotation Age* (T^{SO})
Own stand price	- if $F_T^a \geq 0$ +/- otherwise	- if $F_T^a > 0$, $F_{T\tau}^a > 0$, $F_\tau^a > 0$ *or* if $F_T^a > 0$, $F_{T\tau}^a < 0$, $F_\tau^a < 0$	- *if* $F_T^a \geq 0; F_T^b > 0$ +/- *otherwise*
Own-stand regeneration cost	+	+	+
Adjacent stand price as $F_T^a \geq 0$	+ if $F_{\tau T}^a > 0$ - if $F_{\tau T}^a < 0$	+/-	+ if $F_\tau^b \geq 0; F_\tau^a > 0;$ $F_{\tau T}^a < 0$ - if $F_\tau^b \geq 0; F_\tau^a > 0;$ $F_{\tau T}^a > 0$
Interest rate	- if $F_{\tau T}^a > 0$ +/- otherwise	+/-	- if $F_{T\tau}^b > 0$, $F_T^b > 0$ +/- otherwise
Adjacent stand regeneration cost	+ if $F_{\tau T}^a > 0$ - if $F_{\tau T}^a < 0$	0	+ if $F_{\tau T}^a > 0$ - if $F_{\tau T}^b < 0$

***By symmetry, the comparative statics results in the table would also hold for stand b, with the parameters redefined for this stand.**

Appendix 1. *Proof of Lemma 1*

The proof is given only for $\Omega^{aN}_{T\tau}$, because the proof for $\Omega^{bN}_{\tau T}$ is analogous. The cross-derivative $\Omega^{aN}_{T\tau}$ can be re-expressed as

$$\Omega^{aN}_{T\tau} = \int_0^T F^a_\tau(s,\tau)e^{-rs}ds \left[\frac{F^a_\tau(T,\tau)}{\int_0^T F^a_\tau(s,\tau)e^{-rs}ds} - \frac{r}{1-e^{-rT}} \right] \qquad \text{A.1}$$

- Temporal Independence**:** $F^a_{\tau T} = 0 \;\Rightarrow\; \dfrac{dT}{d\tau} = 0$

Proof. If $F^a_{\tau T} = 0$, $\Omega^{aN}_{T\tau}$ reduces to $\Omega^a_{T\tau} = F^a_\tau(T,\tau) - (1-e^{-rT})^{-1}\left[F^a_\tau(0,\tau) - F^a_\tau(T,\tau)e^{-rT}\right]$. There are two possibilities. If $F^a_\tau = 0$, then trivially $\Omega^{aN}_{T\tau} = 0$. Under $F^a_\tau \neq 0$, $F^a_{\tau T} = 0$ implies that $\left[F^a_\tau(0,\tau) - F^a_\tau(T,\tau)e^{-rT}\right] = F^a_\tau(1-e^{-rT}) \Rightarrow$
$\Omega_{T\tau} = F_\tau(T,\tau) - (1-e^{-rT})^{-1}F_\tau(T,\tau)(1-e^{-rT}) = 0$. Hence, $\dfrac{dT}{d\tau} = 0$.

- Increasing Temporal Dependence: $F^a_{\tau T} > 0 \;\Rightarrow\; \dfrac{dT}{d\tau} > 0$

Proof. i) Assume that $F^a_\tau > 0 \;\Rightarrow\; \Omega^{aN}_{T\tau} > 0 \;\Leftrightarrow\; \dfrac{F^a_\tau(T,\tau)}{\int_0^T F^a_\tau(s,\tau)e^{-rs}ds} > \dfrac{r}{1-e^{-rT}}$.

$F^a_{\tau T} > 0 \;\Rightarrow$

$$\int_0^T F^a_\tau(T,\tau)e^{-rs}ds \;>\int_0^T F^a_\tau(s,\tau)e^{-rs}ds \;\Leftrightarrow\; \frac{F^a_\tau(T,\tau)(1-e^{-rT})}{r} > \int_0^T F^a_\tau(s,\tau)e^{-rs}ds$$

$\Leftrightarrow\; \dfrac{F^a_\tau(T,\tau)}{\int_0^T F^a_\tau(s,\tau)e^{-rs}ds} > \dfrac{r}{1-e^{-rT}}$. Hence, $\Omega^{aN}_{T\tau} > 0$ so that $\dfrac{dT}{d\tau} > 0$.

ii) Assume that $F^a_\tau < 0 \;\Rightarrow\; \Omega^{aN}_{T\tau} < 0 \;\Leftrightarrow\; \dfrac{F^a_\tau(T,\tau)}{\int_0^T F^a_\tau(s,\tau)e^{-rs}ds} < \dfrac{r}{1-e^{-rT}}$.

$F^a_{\tau T} > 0 \;\Rightarrow$

$$\int_0^T F_\tau^a(T,\tau)e^{-rs}ds \quad > \int_0^T F_\tau^a(s,\tau)e^{-rs}ds \Leftrightarrow \frac{F_\tau^a(T,\tau)(1-e^{-rT})}{r} > \int_0^T F_\tau^a(s,\tau)e^{-rs}ds$$

$$\Leftrightarrow \qquad \frac{F_\tau^a(T,\tau)}{\int_0^T F_\tau^a(s,\tau)e^{-rs}ds} < \frac{r}{1-e^{-rT}}$$. Hence, $\Omega_{T\tau}^{aN} > 0$ so that $\frac{dT}{d\tau} > 0$.

- Decreasing Temporal Dependence: $F_{\tau T}^a < 0 \;\Rightarrow\; \frac{dT}{d\tau} < 0$

Proof. i) Assume that $F_\tau^a > 0 \Rightarrow \Omega_{T\tau}^{aN} > 0 \Leftrightarrow \frac{F_\tau^a(T,\tau)}{\int_0^T F_\tau^a(s,\tau)e^{-rs}ds} < \frac{r}{1-e^{-rT}}$.

$F_{\tau T}^a < 0 \Rightarrow$

$$\int_0^T F_\tau^a(T,\tau)e^{-rs}ds \quad < \int_0^T F_\tau^a(s,\tau)e^{-rs}ds \Leftrightarrow \frac{F_\tau^a(T,\tau)(1-e^{-rT})}{r} < \int_0^T F_\tau^a(s,\tau)e^{-rs}ds$$

$$\Leftrightarrow \qquad \frac{F_\tau^a(T,\tau)}{\int_0^T F_\tau^a(s,\tau)e^{-rs}ds} < \frac{r}{1-e^{-rT}}$$. Hence, $\Omega_{T\tau}^{aN} < 0$ so that $\frac{dT}{d\tau} < 0$.

ii) Assume that $F_\tau^a < 0 \;\Rightarrow\; \Omega_{T\tau}^{aN} < 0 \;\Leftrightarrow\; \frac{F_\tau^a(T,\tau)}{\int_0^T F_\tau^a(s,\tau)e^{-rs}ds} > \frac{r}{1-e^{-rT}}$.

$F_{\tau T}^a < 0 \Rightarrow$

$$\int_0^T F_\tau^a(T,\tau)e^{-rs}ds \quad < \int_0^T F_\tau^a(s,\tau)e^{-rs}ds \;\Leftrightarrow\; \frac{F_\tau^a(T,\tau)(1-e^{-rT})}{r} < \int_0^T F_\tau^a(s,\tau)e^{-rs}ds$$

$$\Leftrightarrow \qquad \frac{F_\tau^a(T,\tau)}{\int_0^T F_\tau^a(s,\tau)e^{-rs}ds} > \frac{r}{1-e^{-rT}}$$. Hence, $\Omega_{T\tau}^{aN} < 0$ so that $\frac{dT}{d\tau} < 0$. Q.E.D.

Appendix 2. *Proof of Lemma 2*

Integrating the term $\Psi = \int_0^T F_\tau^a(s,\tau)e^{-rs}ds$ in (8a) by parts and assuming that the third derivative of the amenity function is zero, i.e. $F_{\tau TT}^a = 0$, we get

$$\Psi = \frac{1}{r}\left\{F_\tau^a(0,\tau) - F_\tau^a(T,\tau)e^{-rT} + \frac{1}{r}(F_{\tau T}^a(0,\tau) - F_{\tau T}^a(T,\tau)e^{-rT})\right\} \qquad \text{A2.1}$$

Assume that the amenity valuation function is quadratic

$$F(T,\tau) = \alpha(T+\tau) - \tfrac{1}{2}\beta(T+\gamma\tau)^2, \qquad \text{A2.2}$$

with $\beta > 0$, $\gamma > (<)\, 0$, so that $F_\tau(T,\tau) = \alpha - \beta\gamma(T+\gamma\tau)$, and $F_{\tau T}(T,\tau) = -\beta\gamma$. Moreover, we assume that

$$F_\tau(0,\tau) = \alpha - \beta\gamma^2\tau \begin{Bmatrix} > \\ < \end{Bmatrix} 0 \text{ as } \begin{Bmatrix} \text{stands are complements} \\ \text{stands are substitutes} \end{Bmatrix} \qquad \text{A2.3}$$

Hence, we can express A2.1 as

$$\Psi = \frac{1}{r}\left\{(\alpha - \beta\gamma^2\tau)(1-e^{-rT}) + \beta\gamma T e^{-rT} - \frac{\beta\gamma(1-e^{-rT})}{r}\right\} \qquad \text{A2.4}$$

Using second-order approximation $e^{-rT} = \dfrac{1}{1 + rT + (1/2)r^2T^2}$ yields

$$\Psi = \frac{(1-e^{-rT})}{r}\left\{(\alpha - \beta\gamma^2\tau) - \beta\gamma\frac{T}{2+rT}\right\} = \frac{(1-e^{-rT})}{r}\left\{F_\tau^a(0,\tau) - \beta\gamma\frac{T}{2+rT}\right\} \qquad \text{A2.5}$$

Now,

- For complements $\gamma < 0$, and under increasing temporal dependence $F_\tau^a(T,\tau) = \alpha - \beta\gamma^2\tau - \beta\gamma T > 0$, while decreasing temporal dependence implies that $F_\tau^a(T,\tau) = \alpha - \beta\gamma^2\tau - \beta\gamma T < 0$. Thus A2.5 is positive in both cases: the first one automatically, the second one, because $T > \dfrac{T}{2+rT}$.
- For substitutes $\gamma > 0$. Thus A2.5 is automatically negative for decreasing temporal dependence; and negative also under increasing temporal dependence, because $T > \dfrac{T}{2+rT}$.

QED.

Appendix 3: ***Second order conditions for the sole owner***

The second order conditions rely on the following derivatives, from (10a) and (10b),

$$W_{TT} = -\frac{e^{-rT}}{(1-e^{-rT})}\left[\frac{r}{(1-e^{-rT})}\Omega_T^{aN} - \Omega_{TT}^{aN}\right] + \frac{\int_0^\tau F_{TT}^b(T,x)e^{-rx}dx}{(1-e^{-r\tau})} < 0$$

$$W_{\tau\tau} = -\frac{e^{-r\tau}}{(1-e^{-r\tau})}\left[\frac{r}{(1-e^{-r\tau})}\Omega_{\tau}^{bN} - \Omega_{\tau\tau}^{bN}\right] + \frac{\int_0^T F_{\tau\tau}^{a}(s,\tau)e^{-rs}ds}{(1-e^{-rT})} < 0$$

$$\Delta^{SO} = W_{TT}W_{\tau\tau} - W_{T\tau}W_{\tau T} > 0\,,$$

Appendix 4. *Comparative statics of adjacent stand's timber price and interest rate in Stackelberg model*

- *Price (q):*

By differentiation we get

$$\Omega_{Tq}^{aS} = \tau_q(.)\left[F_{\tau}(T,\tau)e^{-rT} - \frac{re^{-rT}}{(1-e^{-rT})}\int_0^T F_{\tau}(s,\tau)e^{-rs}ds + \tau_T(.)\int_0^T F_{\tau\tau}(s,\tau)e^{-rs}ds\right] \qquad \text{A4.1}$$

$$+\tau_{Tq}(.)\int_0^T F_{\tau}(s,\tau)e^{-rs}ds\,,$$

where the term τ_{Tq} is defined as follows:

$$\tau_{Tq} = \frac{F_T^b - r(1-e^{-r\tau})^{-1}\int_0^{\tau} F_T^b e^{-rx}dx)(g''(\tau) - rg'(\tau))}{(\Omega_{\tau\tau}^{bS})^2} \begin{Bmatrix} > \\ = \\ > \end{Bmatrix} 0 \;\; as \;\; F_{T\tau}^b \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0\,. \qquad \text{A4.2}$$

Thus Ω_{Tq}^{aS} consists of a shift in the follower's reaction function as well as a change in its slope. The sign of all terms depends on the nature of temporal dependence between the stands. On the basis of our previous analysis the sum of the first two terms of eqn A3.1 in brackets are negative (positive) for increasing (decreasing) temporal dependence. The third and fourth terms are of opposite sign, being positive (negative) for increasing (decreasing) temporal dependence, indicating that the sign of Ω_{Tq}^{aS} is a priori ambiguous for both decreasing and increasing temporal dependence. For temporally independent stands we have $\Omega_{Tq}^{aS} = 0$.

- *Interest rate (r):*

We first develop the cross-derivative of the follower's reaction function, $\tau_{Tr}(.)$. Recalling equation (8b), the derivative with respect to r is,

$$\tau_{Tr} = \frac{1}{\phi^2}\left\{F_T^b - r(1-e^{-r\tau})^{-1}\int_0^T F_T^b e^{-rx}dx + \phi\left[r(1-e^{-r\tau})^{-1}\int_0^T xF_T^b e^{-rx}dx - \omega\int_0^T F_T^b e^{-rx}dx\right]\right\},$$

A4.3.

where $\phi = qg''(\tau) - rqg'(\tau) + F_\tau^b < 0$ and $\omega = \dfrac{1}{1-e^{-r\tau}}\left(1 - \dfrac{\tau r e^{-r\tau}}{1-e^{-r\tau}}\right)$.

The sign of the first two terms in braces depends on the sign of $F_{T\tau}^b$ by Lemma 1. Approximating the ω term using $e^{-r\tau} = \dfrac{1}{1 + r\tau + (1/2)r^2\tau^2}$ implies that $\omega = \dfrac{1}{1-e^{-r\tau})}\left[\dfrac{\frac{1}{2}r^2\tau^2}{1 + r\tau + (1/2)r^2\tau^2}\right] > 0$. We determine the sign of the integral term $\int_0^T xF_T^b e^{-rx}dx$ by using the quadratic amenity valuation function given in A.2.2 in Lemma 3.

Lemma 3. *If the amenity valuation function is quadratic, then*

$$\int_0^\tau xF_T^b e^{-rx}dx \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \text{ as } F_T^b(T,0 \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0.$$

Proof. *See Appendix 5.*

Hence, we have established that $\tau_{Tr} \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0$ as $F_{\tau T}^a \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0$. Now the overall sign of the three latter terms in Ω_{Tr}^{aS} can be revealed. The first term is positive, but the last two terms are negative when the stands exhibit increasing temporal dependence, while the opposite holds for decreasing temporal dependence. Hence, the interest rate effect in the Stackelberg game is genuinely ambiguous.

Appendix 5. *Proof of Lemma 3*

Integrating $\int_0^\tau xF_T^b e^{-rx}dx$ by parts and assuming that the third derivative of the amenity function is zero, i.e. that $F_{T\tau\tau} = 0$, we get

$$\int_0^\tau xF_T^b e^{-rx}dx = \int_0^\tau F_T^b e^{-rx}dx + \int_0^\tau xF_{T\tau}^b e^{-rx}dx \qquad \text{A5.1}$$

Assume the quadratic amenity valuation function

$$F(T,\tau) = \alpha(T+\tau) - \tfrac{1}{2}\beta(T+\gamma\tau)^2, \qquad \text{A5.2}$$

Applying this to A5.1 yields $\int_0^\tau xF_T^b e^{-rx} dx = \int_0^\tau (\alpha - \beta T - \beta\gamma x) x e^{-rx} dx$. Noting that the term $(\alpha - \beta T)$ is independent of the integral we have

$$\int_0^\tau xF_T^b e^{-rx} dx = (\alpha - \beta T)\int_0^\tau x e^{-rx} dx - \beta\gamma \int_0^\tau x^2 e^{-rx} dx \qquad \text{A5.3}$$

By integrating the first RHS of A5.3 one gets $\int_0^\tau x e^{-rx} dx = -\frac{1}{r}\left[\tau e^{-r\tau} - \frac{1}{r}(1 - e^{-r\tau})\right]$ and using second-order approximation $e^{-rT} = \frac{1}{1 + rT + (1/2)r^2T^2}$ yields

$$\int_0^\tau x e^{-rx} dx = \frac{\tau^2}{2r(1 + r\tau + \frac{1}{2}r^2\tau^2)} > 0, \qquad \text{A5.4}$$

For the second term we get via integration by parts that

$$-\beta\gamma\int_0^\tau x^2 e^{-rx} dx = -\beta\gamma\left\{-\frac{\tau^2}{r(1 + r\tau + \frac{1}{2}r^2\tau^2)} + \frac{2}{r}\frac{\tau^2}{2(1 + r\tau + \frac{1}{2}r^2\tau^2)}\right\} = 0 \quad \text{A5.5}$$

Given the positivity of A4.4, the sign of the integral $\int_0^\tau xF_T^b e^{-rx} dx$ depends on the sign of $(\alpha - \beta T)$. Hence, we have shown that

$$\int_0^\tau xF_T^b e^{-rx} dx \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \text{ as } F_T^b(T,0) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0. \qquad \text{A5.6}$$

QED.

Appendix 6.

Given (21a) and (21b) in the text, the first terms in W_{Tr} and $W_{\tau r}$ are negative. Next we study sign of W_{Tr} brace terms (by symmetry this holds also for the sign of the brace terms in $W_{\tau r}$). According to Lemma 2 we have

$$\int_0^{\tau} F_T^b(T,x)e^{-rx}dx = \frac{(1-e^{-r\tau})}{r}\left\{(\alpha - \beta T) - \beta\gamma\,\frac{\tau}{2+r\tau}\right\} =$$

$$\frac{(1-e^{-r\tau})}{r}\left\{F_T^b(T,0) - \beta\gamma\,\frac{\tau}{2+r\tau}\right\} \qquad \text{A6.7}$$

where $1-e^{-r\tau} = (r\tau + (1/2)r^2\tau^2)/(1+r\tau + (1/2)r^2\tau^{2\prime})$. Now we have

$$(1-e^{-rT})\int_0^{\tau} F_T^b(T,x)e^{-rx}dx = \frac{\left[rT + (\frac{1}{2})r^2T^2\right]\left[r\tau + (\frac{1}{2})r^2\tau^2\right]}{r\left[1+rT + (\frac{1}{2})r^2T^2\right]\left[1+r\tau + (\frac{1}{2})r^2\tau^2\right]}\left\{F_T^b(T,0) - \beta\gamma\,\frac{\tau}{2+r\tau}\right\}$$

A6.8

Using Lemma 3 we have

$$\int_0^{\tau} xF_T^b e^{-rx}dx = (\alpha - \beta T)\frac{\tau^2}{2r(1+r\tau + (1/2)r^2\tau^2)} = F_T^b(T,0)\frac{\tau^2}{2r(1+r\tau + (1/2)r^2\tau^2)}$$

A6.9

Combining A6.8. and A6.9 gives

$$(1-e^{-rT})\int_0^{\tau} F_T^b(T,x)e^{-rx}dx - \frac{1}{T}\int_0^{\tau} xF_T^b(t,x)e^{-rx}dx =$$

$$\frac{1}{r\left[1+r\tau + (\frac{1}{2})r^2\tau^2\right]}\left\{\frac{\left[rT + (\frac{1}{2})r^2T^2\right]\left[r\tau + (\frac{1}{2})r^2\tau^2\right]}{\left[1+rT + (\frac{1}{2})r^2T^2\right]}F_T^b(T,0) - \beta\gamma\,\frac{\tau}{2+r\tau} - \frac{\tau^2}{2T}F_T^b(T,0)\right\}$$

A4.10

Analogously we can write

$$(1-e^{-rT})\int_0^{\tau} F_T^b(T,x)e^{-rx}dx - \frac{1}{T}\int_0^{\tau} xF_T^b(t,x)e^{-rx}dx =$$

$$\frac{1}{r\left[1+rT+(\frac{1}{2})r^2T^2\right]}\left\{\frac{\left[rT+(\frac{1}{2})r^2T^2\right]\left[r\tau+(\frac{1}{2})r^2\tau^2\right]}{\left[1+r\tau+(\frac{1}{2})r^2\tau^2\right]}F_\tau^a(0.\tau)-\beta\gamma\frac{\tau}{2+r\tau}-\frac{T^2}{2\tau}F_\tau^a(0.\tau)\right\}$$

A6.11

Under plausible assumptions concerning the interest rate and rotation ages the last term in equations A6.10 and A6.11 dominates.Q.E.D.