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EXPECTATIONAL STABILITY OF RESONANT FREQUENCY SUNSPOT EQUILIBRIA

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Abstract

We consider the stability under adaptive learning of the complete set of solutions to the basic linear forward looking model in which the current value of the state variable depends linearly on the (subjectively) expected value of the state next period and the coefficient of the expected state is bigger than one in absolute value. In addition to the fundamentals solution, the literature describes both finite-state Markov sunspot solutions, satisfying a resonant frequency condition, and autoregressive solutions depending on an arbitrary martingale difference sequence. We clarify the relationships between these solutions and show that the stability properties of equilibria may depend crucially on the representation used by agents in the learning process. Only the finite-state Markov sunspot solutions can be stable under learning.

Keywords: Indeterminacy, representations of solutions, learnability, expectational stability, endogenous fluctuations.

JEL Classification: C62, D83 D84, E31, E32.

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Expectational Stability of Resonant Frequency Sunspot Equilibria*

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Abstract

We consider the stability under adaptive learning of the complete set of solutions to the model $x_t = \beta E_t^* x_{t+1}$ when $|\beta| > 1$. In addition to the fundamentals solution, the literature describes both finite-state Markov sunspot solutions, satisfying a resonant frequency condition, and autoregressive solutions depending on an arbitrary martingale difference sequence. We clarify the relationships between these solutions and show that the stability properties of equilibria may depend crucially on the representation used by agents in the learning process. Only the finite-state Markov sunspot solutions can be stable under learning.

JEL classifications: C62, D83, D84, E31, E32

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1 Introduction

We consider a univariate linear model of the form

$$x_t = \beta E_t^* x_{t+1}, \tag{1}$$

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both under rational expectations and under the assumption that economic agents follow certain natural adaptive learning rules. Here $E_t^* x_{t+1}$ denotes the expectations held by agents, assumed homogeneous. Under rational expectations

$$x_t = \beta E_t x_{t+1}, \tag{2}$$

where $E_t x_{t+1}$ denotes the true conditional expectation given the information set at time t. We are interested in whether rational expectations solutions depending on extraneous exogenous random variables (sunspots) can be stable under adaptive learning.

The model (1) does not include intrinsic random shocks, and has been centered, so that there is a steady state at the origin. Both of these assumptions are made for simplicity and could be relaxed. Throughout the paper we will assume that $\beta \neq 0$ and $\beta \neq 1$.

The steady state $x_t = 0$ is a rational expectations (RE) solution (and indeed satisfies perfect foresight). This solution is often called the "fundamentals" solution. As is well known, there are other rational expectations solutions to (1), taking the form

$$x_t = \beta^{-1} x_{t-1} + \varepsilon_t, \tag{3}$$

where ε_t is an arbitrary martingale difference sequence, i.e. a stochastic process satisfying $E_t \varepsilon_{t+1} = 0$. It is easily verified that a process of the form (3) is indeed a RE solution to (1). Conversely, any RE solution to (1) can be written in the form (3), as can be seen by defining $\varepsilon_{t+1} = x_{t+1} - E_t x_{t+1}$. We will refer to (3) as the $AR(1)$ representation of the solution, as we now explain.

When $|\beta| < 1$ the solution $x_t \equiv 0$ is the unique nonexplosive solution, see (Gourieroux, Laffont, and Monfort 1982). The other rational expectations solutions of the form (3) have conditional expectations that, in absolute value, tend to infinity. In the "irregular" case $|\beta| > 1$ there are multiple stationary solutions. In particular, if ε_t is an *iid* process with mean 0 and constant variance (i.e. "white noise"), then (3) is a stationary first-order autoregressive (or $AR(1)$) process.[1] When ε_t takes a different form the process need not be

[1] If the system begins at $t = 0$ then the solution (3) also requires an initial condition. For an appropriate initial condition the process is stochastically stationary. For other initial conditions there is a transient nonstationarity, but the process is asymptotically stationary.

stationary, but for convenience we will continue to refer to (3) as the $AR(1)$ representation of the solution.

Considerable attention is given in the literature to finite-state Markov solutions, i.e. solutions taking the form

$$x_t = \bar{x}_i \text{ when } s_t = i, \text{ for } i = 1, \ldots, K, \tag{4}$$

where $s_t \in \{1, \ldots, K\}$ follows a finite-state Markov process with fixed transition probabilities $\pi_{ij} = P(s_{t+1} = j \mid s_t = i)$. Again, the exogenous stationary process s_t is usually called a "sunspot" process and the solutions are called finite-state Markov stationary sunspot equilibria (SSEs).

Much of this literature considers such solutions more generally in the context of possibly nonlinear models, i.e. in models of the form $x_t = E_t^* F(x_{t+1})$. For example, the existence question is discussed at length in (Guesnerie and Woodford 1992) and (Chiappori, Geoffard, and Guesnerie 1992). However, it can easily be established that this type of SSE exists in the linear model when $|\beta| > 1$. Clearly, such solutions must have an equivalent representation in the $AR(1)$ form (3).

The learning question has also been considered for both forms of solution (3) and (4). In particular, for solutions in the $AR(1)$ form, stability under adaptive learning was considered in (Evans 1989) and discussed further in Section 9.7 of Chapter 9 of (Evans and Honkapohja 2001b). Stability under adaptive learning of Markov SSEs is discussed in (Woodford 1990), (Evans 1989), (Evans and Honkapohja 1994) and Chapter 12 of (Evans and Honkapohja 2001b).[2]

There are some important gaps in this literature, even in the linear case. The relationship between these two types of SSEs has not been explicitly addressed, and existing adaptive learning results are incomplete. Indeed, the adaptive learning results for the two set-ups appear to be at variance, as we show below. In this paper we clarify the relationship between the different solutions and then extend and reconcile the learning results by nesting the two set-ups in a common framework.

[2]For a study of the eductive stability of 2-state Markov SSEs see (Desgranges and Negroni 2000).

2 Resonant Frequency Sunspots

We now consider finite-state Markov SSEs. For convenience we focus on the 2-state case, which has a prominent role in the literature. Assuming solutions to (2) of the form (4) implies

$$\begin{aligned}\bar{x}_1 &= \beta(\pi_{11}\bar{x}_1 + \pi_{12}\bar{x}_2),\\ \bar{x}_2 &= \beta(\pi_{21}\bar{x}_1 + \pi_{22}\bar{x}_2),\end{aligned}$$

or

$$\Pi\bar{x} = \beta^{-1}\bar{x}, \tag{5}$$

where Π is the matrix with elements π_{ij} and $\bar{x}' = (\bar{x}_1, \bar{x}_2)$. In order for the solution to constitute an SSE we must, of course, have $\bar{x}_1 \neq \bar{x}_2$. It follows that β^{-1} must be an eigenvalue of Π with eigenvector $\bar{x}$. Since the eigenvalues of Π are 1 and $\pi_{11} + \pi_{22} - 1$ we obtain

$$\pi_{11} + \pi_{22} - 1 = \beta^{-1} \tag{6}$$

$$(1 - \pi_{22})\bar{x}_1 + (1 - \pi_{11})\bar{x}_2 = 0. \tag{7}$$

It is seen from (5) or (7) that there exists a one-dimensional continuum of $(\bar{x}_1, \bar{x}_2)$ for transition probabilities satisfying (6).[3]

We will refer to the sunspot variables that satisfy (6) as "resonant frequency" sunspots, since only sunspots precisely satisfying the transition probability restriction (6) are capable of generating finite-state sunspot fluctuations around the steady-state.

From Section 1 we know that it is also possible to represent such *resonant frequency stationary sunspot equilibria* (RFSSE) in the form (3). Requiring that the RFSSE $(\bar{x}_1, \bar{x}_2)$ is of the form (3), in which ε_t is a linear function of s_t and s_{t-1} and in which the transition probabilities for s_t satisfy (6), implies that the corresponding martingale difference sequence ε_t can be written as

$$\varepsilon_t = \bar{a} + \bar{f}s_t + \bar{g}s_{t-1}, \text{ where} \tag{8}$$

$$\begin{aligned}\bar{f} &= \bar{x}_2 - \bar{x}_1,\ \bar{g} = (1 - \pi_{11} - \pi_{22})(\bar{x}_2 - \bar{x}_1) \text{ and} \qquad (9)\\ \bar{a} &= -(3 - 2\pi_{11} - \pi_{22})(\bar{x}_2 - \bar{x}_1).\end{aligned}$$

[3] Corresponding results for the K-state SSEs are given in (Chiappori, Geoffard, and Guesnerie 1992).

It can be verified that $E_t\varepsilon_{t+1} = 0$. Note, however, that ε_t is not an *iid* process. Note also that the non-uniqueness of $(\bar{x}_1, \bar{x}_2)$ corresponds to a degree of freedom in the parameters $\bar{a}$, $\bar{f}$ and $\bar{g}$.

The RFSSE is obtained from the general representation (3) with this choice of ε_t, provided the initial condition is $\bar{x}_1$ or $\bar{x}_2$. For other choices of initial condition, the process (3), with ε_t given by (8)-(9), is an asymptotically stationary solution, tending in the limit to the RFSSE.

3 E-stability and Adaptive Learning

3.1 E-Stability of $AR(1)$ Sunspot Equilibria

We now take up the question of the local stability under learning of SSEs. Consider first the general representation for SSEs (3). Suppose agents believe that x_t follows an $AR(1)$ process, but are not certain of the coefficients, which they estimate and revise over time. More particularly, suppose u_t is an observable process that satisfies $E_t u_{t+1} = 0$. Suppose that agents believe that x_t follows the process

$$x_t = a + bx_{t-1} + cu_t + \eta_t,$$

where b and c are unknown fixed parameters and η_t represents an unobserved white noise disturbance that agents might believe to be present. In an SSE $a = 0$, $b = \beta^{-1}$, $\eta_t \equiv 0$ and c is arbitrary (in the earlier notation $\varepsilon_t = cu_t$). The fundamentals solution is given by $a = b = c = 0$ and $\eta_t \equiv 0$.

The standard way to formulate adaptive learning in this context is "least squares learning," in which agents at time t estimate a, b and c by a least-squares regression of x_i on x_{i-1}, u_i and an intercept, using data $i = 1, \ldots, t-1$. Note that estimates are updated each period. Assuming that the time t information set is $I_t = \{u_t, u_{t-1}, \ldots, x_{t-1}, x_{t-2}, \ldots\}$, the forecasts $E_t^* x_{t+1}$ are obtained by taking the conditional expectation of $x_{t+1} = a + b(a + bx_{t-1} + cu_t + \eta_t) + cu_{t+1} + \eta_{t+1}$. This yields

$$E_t^* x_{t+1} = a(1+b) + b^2 x_{t-1} + bcu_t, \tag{10}$$

where a, b and c are replaced by their current least-squares estimates a_t, b_t and c_t.

Stability of a rational expectations equilibrium under least-squares learning is defined in terms of the dynamic system given by the exogenous sunspot process u_t and by (1), where $E_t^* x_{t+1} = a_t(1+b_t)+b_t^2 x_{t-1}+b_t c_t u_t$ and (a_t, b_t, c_t) are updated by least squares. If $(a_t, b_t, c_t) \to (0, 0, 0)$ then the fundamentals solution is said to be stable, while if $(a_t, b_t, c_t) \to (0, \beta^{-1}, c)$ for some c then the class of SSEs is said to be stable. Here local stability is the relevant concept and we omit precise definitions of the appropriate notions of stochastic convergence. For details see (Evans and Honkapohja 2001b).

For a wide range of economic models it has been shown that stability under least-squares learning is governed by expectational stability (E-stability), and in this paper we therefore focus on determining the conditions for the various solutions to be E-stable. E-stability is defined in terms of the mapping from the Perceived Law of Motion (PLM), parameterized here by (a, b, c), to the implied parameters $T(a, b, c)$ of the Actual Law of Motion (ALM). The ALM parameters, corresponding to a given PLM, are here obtained by inserting the corresponding expectation rule (10) into the model (1), yielding

$$T(a, b, c) = (\beta a(1 + b), \beta b^2, \beta bc).$$

Note that the fixed points of the T map correspond to the fundamentals solution $(0, 0, 0)$ and to the continua of SSEs $(0, \beta^{-1}, c)$.

E-stability of an REE (or a set of REE) is determined by local stability of the REE (or a set of REE) under the ordinary differential equation

$$\frac{d}{d\tau}(a, b, c) = T(a, b, c) - (a, b, c),$$

where τ denotes notional or virtual time. Since the Jacobian of the right-hand side, evaluated at $(0, 0, 0)$ has one eigenvalue of $\beta - 1$ and two eigenvalues of -1, it follows that the fundamentals solution is E-stable provided $\beta < 1$. On the other hand the set of SSEs are not E-stable. This can be seen from the differential equation for b, which is given by $db/d\tau = \beta b^2 - b$ and which is always locally unstable at $b = \beta^{-1}$. If $\beta > 1$, none of the solutions are E-stable. We summarize these observations in the following:

Proposition 1 *The set of SSEs of the form (3) is not E-stable. The fundamental solution is E-stable if $\beta < 1$ and is not E-stable if $\beta > 1$.*[4]

This way of looking at the full set of rational expectations solutions to (1) thus clearly favors the fundamentals solution. Provided $\beta < 1$, the fundamentals solution is E-stable, and hence locally stable under least-squares

[4]These results were first obtained in (Evans 1989).

learning, while the set of $AR(1)$ SSEs is never E-stable and hence is locally unstable under least-squares learning.

3.2 E-stability of Resonant Frequency Sunspot Solutions

We turn now to the stability of RFSSEs, i.e. to the 2-state Markov sunspot equilibria given by (6)-(7). The exogenous sunspot variable s_t is assumed to be observable at t, with known transition probabilities π_{11}, π_{22}. We assume that agents do not know the values $\bar{x}_1, \bar{x}_2$, taken in the RFSSE, and that they therefore estimate their values. A simple and natural adaptive learning rule is state contingent averaging, i.e. $\bar{x}_j$, $j = 1, 2$, is estimated to be the average of the values for x_t obtained when $s_t = j$.[5]

(Evans and Honkapohja 1994) and Chapter 12 of (Evans and Honkapohja 2001b) shows how E-stability governs local convergence of adaptive learning to finite-state SSEs. Following the E-stability principle we look at the mapping from the PLM to the ALM. The PLM is now

$$x_t = x_j + \eta_t \text{ if } s_t = j, \text{ for } j = 1, 2,$$

where in an RFSSE $x_j = \bar{x}_j$ and $\eta_t \equiv 0$. In state $s_t = 1$ the expectation corresponding to this PLM is $E_t^* x_{t+1} = \pi_{11} x_1 + (1 - \pi_{11}) x_2$ and in state $s_t = 2$ we have $E_t^* x_{t+1} = (1 - \pi_{22}) x_1 + \pi_{22} x_2$. Inserting these into (1) yields

$$\begin{aligned} x_t &= \beta \pi_{11} x_1 + \beta (1 - \pi_{11}) x_2 \text{ if } s_t = 1, \text{ and} \\ x_t &= \beta (1 - \pi_{22}) x_1 + \beta \pi_{22} x_2 \text{ if } s_t = 2. \end{aligned}$$

The corresponding map from PLM $x = (x_1, x_2)'$ to ALM $T(x)$ is given by

$$T(x) = \beta \Pi x \text{ or } T(x) = (\pi_{11} + \pi_{22} - 1)^{-1} \Pi x,$$

where we have used the condition (6). The E-stability condition is given by local stability of the set of equilibria of $dx/d\tau = T(x) - x$. The fixed points of T, i.e. the equilibria of the differential equation, are given by (7). This set forms a 1-dimensional continuum. The eigenvalues of $(\pi_{11} + \pi_{22} - 1)^{-1} \Pi - I$ are given by 0 and $(\pi_{11} + \pi_{22} - 1)^{-1} - 1$. It follows that the continuum is

[5] This formulation was suggested in (Evans and Honkapohja 1994). An alternative and essentially equivalent prodedure (see the following section) would be for agents to estimate a least squares regression of x_t on s_t and an intercept.

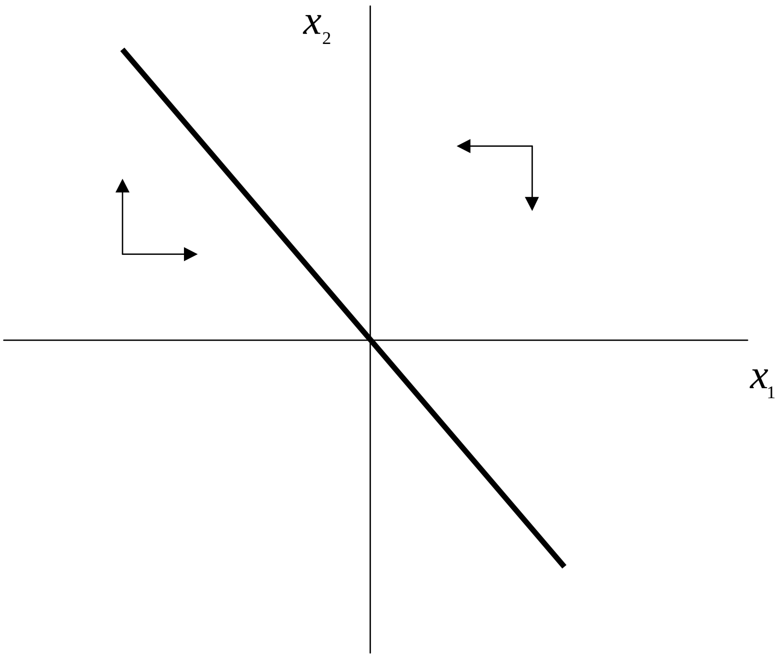

Figure 1: E-stable continuum when $\pi_{11} + \pi_{22} < 1$

E-stable if $\pi_{11} + \pi_{22} < 1$ and not E-stable if $\pi_{11} + \pi_{22} > 1$. In terms of the model parameter β we can thus state:

Proposition 2 *The set of RFSSEs is E-stable if $\beta < -1$ and it is not E-stable if $\beta > 1$.*

The accompanying Figure illustrates E-stability for the case $\beta < -1$. The result that an RFSSE is not E-stable when $\beta > 1$ was previously given in (Evans and Honkapohja 1994), but E-stability of the set of RFSSEs when $\beta < -1$ has not been previously noted.

4 E-stability in a General Framework

The preceding section appears at first sight to provide incompatible results concerning the adaptive stability of RFSSEs. Section 3.1 shows the lack of stability under learning of all sunspot equilibria, while 3.2 shows the stability under learning of certain resonant frequency sunspot equilibria. To understand the relation between these results, we now examine a class of PLMs that nests the RFSSEs in a way that includes $AR(1)$ representations.

Thus, consider the class of PLMs taking the form

$$x_t = a + bx_{t-1} + fs_t + gs_{t-1} + \eta_t,$$

where η_t is assumed to be white noise. Under rational expectations $\eta_t \equiv 0$ and the coefficients a, b, f, g satisfy certain restrictions. We do not yet make the assumption that the sunspot process s_t satisfies the resonant frequency condition (6). We also do not yet impose any condition on β, other than $\beta \neq 0$.

For this PLM the corresponding $E_t^* x_{t+1}$ is obtained from

$$x_{t+1} = a + b(a + bx_{t-1} + fs_t + gs_{t-1} + \eta_t) + fs_{t+1} + gs_t + \eta_{t+1}$$

by taking conditional expectations. This yields

$$E_t^* x_{t+1} = a(1+b) + b^2 x_{t-1} + fE_t s_{t+1} + (bf + g)s_t + bgs_{t-1}.$$

Expressing $E_t s_{t+1}$ as a linear function of s_t, we have

$$E_t s_{t+1} = (3 - 2\pi_{11} - \pi_{22}) + (\pi_{11} + \pi_{22} - 1)s_t.$$

Inserting these into (1) yields the following map from PLM to ALM:

$$\begin{aligned} & T(a, b, f, g) \\ = \; & \left(\beta a(1+b) + \beta f(3 - 2\pi_{11} - \pi_{22}), \beta b^2, \beta f(\pi_{11} + \pi_{22} - 1 + b) + \beta g, \beta bg\right). \end{aligned}$$

Rational expectations solutions are given by the fixed points of this map (together with $\eta_t \equiv 0$). There are three classes of solutions as specified by the following proposition:

Proposition 3 *The rational expectations solutions can be divided into three classes:*
(I) $a = b = f = g = 0$*. This is the "fundamentals" solution.*
(II) When (6) holds, there are solutions of the form $b = g = 0$*, with* f *arbitrary and* $a = \frac{3-2\pi_{11}-\pi_{22}}{\pi_{11}+\pi_{22}-2} f$*. For* $f \neq 0$ *we obtain the RFSSEs.*[6]

[6] $f = 0$ yields the fundamentals solution as a special case.

(III) Setting $b = \beta^{-1}$, f arbitrary, $g = -(\pi_{11} + \pi_{22} - 1)f$ and $a = -(3 - 2\pi_{11} - \pi_{22})f$ yields the set of $AR(1)$ solutions generated by choices of ε_t as a linear combination of s_t, s_{t-1} and an intercept.[7]

Consider now E-stability for these three sets of rational expectations solutions. For E-stability we require local stability of the solution set under the differential equation

$$d/d\tau(a, b, f, g) = T(a, b, f, g) - (a, b, f, g).$$

The relevant condition is that all roots of the Jacobian $DT - I$ have negative real parts. For the fundamentals solution (I) this leads to the conditions $\beta < 1$ and $\beta(\pi_{11} + \pi_{22} - 1) < 1$. These conditions are never satisfied if $\beta > 1$, are always satisfied if $|\beta| < 1$ and may be satisfied if $\beta < -1$. (However, note that they will not be satisfied if $\beta < -1$ and (6) holds). Next, we consider the $AR(1)$ solution set (III). The differential equation for b is autonomous and is always unstable at $b = \beta^{-1}$. Thus this solution set is never E-stable.

Lastly, consider the solution set (II). These exist only when $|\beta| > 1$ and the resonant frequency condition (6) holds. The subsystem in b, g is autonomous and $b = g = 0$ is always locally stable. It can then be verified that the condition for local stability condition of this solution set is that is that $\beta < 1$. Thus the set of RFSSEs is not E-stable if $\beta > 1$ but is E-stable if $\beta < -1$.

We collect these arguments in the following proposition:

Proposition 4 *(a) The fundamental solution (I) is E-stable if $\beta < 1$ and $\beta(\pi_{11} + \pi_{22} - 1) < 1$ and is not E-stable if $\beta > 1$ or $\beta(\pi_{11} + \pi_{22} - 1) > 1$.*
(b) The solution set (II) of RFSSEs, which exists only when $|\beta| > 1$ and (6) holds, is E-stable if $\beta < -1$ and it is not E-stable if $\beta > 1$.
(c) The solution set (III) is never E-stable whether or not condition (6) holds.

To summarize, when $|\beta| < 1$, stationary sunspot equilibria do not exist. The fundamentals solution is E-stable, and therefore stable under adaptive learning.[8] There exist explosive sunspot equilibria, taking the $AR(1)$ form,

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[7] Imposing (6) and an appropriate initial condition gives the $AR(1)$ representation of the RFSSEs.

[8] We remark that the conditions in part (a) of the proposition are "strong E-stability" conditions for the fundamental solution, i.e. the stability conditions required when the PLM allows for the presense of sunspots. The weaker condition $\beta < 1$ is sufficient for E-stability of the fundamental solution when the PLM does not include a possible dependence on a sunspot.

but these are not E-stable. When $\beta > 1$ SSEs (stationary sunspot equilibria) do exist, but no solution or solution set is E-stable. When $\beta < -1$ but the resonant frequency condition does not hold, SSEs and asymptotically stationary sunspot equilibria do exist, taking the $AR(1)$ from, but they are not E-stable. Finally, when $\beta < -1$ and the resonant frequency condition (6) holds, RFSSEs exist and are E-stable. There are two ways to represent the same solution, but the $AR(1)$ representation is not E-stable. Adaptive learning will, however, locally converge to the RFSSE if the conditions $\beta < -1$ and $\pi_{11} + \pi_{22} - 1 = \beta^{-1}$ are satisfied.

We conclude by noting that the RFSSEs are a kind of "common factor" solution similar to those discussed in (Evans and Honkapohja 1986). Solutions of type (III) are of the form

$$x_t = -(3 - 2\pi_{11} - \pi_{22})f + \beta^{-1}x_{t-1} + fs_t - (\pi_{11} + \pi_{22} - 1)fs_{t-1}.$$

Under the resonant frequency condition (6) we obtain

$$(1 - \beta^{-1}L)x_t = -f(3 - 2\pi_{11} - \pi_{22}) + f(1 - \beta^{-1}L)s_t,$$

where L is the lag operator defined by $Lx_t = x_{t-1}$. The two sides of this equation have the lag polynomial $1 - \beta^{-1}L$ as a common factor. For an appropriate initial condition on x_t the stochastic process is stationary and we can multiply through by $(1 - \beta^{-1}L)^{-1}$ to cancel this common factor and obtain

$$x_t = \frac{-f(3 - 2\pi_{11} - \pi_{22})}{1 - \beta^{-1}} + fs_t,$$

which is indeed the form of the RFSSE.

It has been observed in previous work, for example see Chapters 8 and 9 of (Evans and Honkapohja 2001b) and the references cited in (Evans and Honkapohja 1999), that common factor solutions can have different stability properties under adaptive learning from those of the larger set of solutions in which they are located. From this viewpoint the apparently contradictory results of the previous section are not surprising. Convergence to an RFSSE can arise only when the PLM is parameterized in such a way that it includes the common factor representation of the RFSSE.

5 Conclusion

We've considered the simplest dynamic expectations model which permit stationary sunspot equilibria. In the linear one-step forward looking model $x_t = \beta E_t^* x_{t+1}$, SSEs exist when $|\beta| > 1$. These can take the form of $AR(1)$ solutions or of finite-state Markov processes. The latter must satisfy a "resonant frequency" restriction on the transition probabilities. In line with the previous literature we have found that none of these SSEs are stable under adaptive learning when $\beta > 1$. However, when $\beta < -1$, two-state Markov SSEs are stable under adaptive learning, even though the whole class of SSEs taking the $AR(1)$ form is not. Our two main conclusions are thus that stability under adaptive learning of SSEs requires $\beta < -1$, and that in this case only a subset of the SSEs, namely the resonant frequency sunspots, are stable.

One might wonder how sensitive are the results to the assumed linearity of the model. In a companion paper (Evans and Honkapohja 2001a) we show how our results carry over to the corresponding nonlinear model $x_t = E_t^* F(x_{t+1})$ in a neighborhood of a steady state $\hat{x}$. In particular, if $F'(\hat{x}) > 1$ there are no nearby E-stable SSEs. In contrast, if $F'(\hat{x}) < -1$ there exist E-stable SSEs in every neighborhood of the steady state.

The linear and nonlinear models do differ in the following respect. In the linear case two-state Markov SSEs must satisfy exactly the resonant frequency condition $\beta^{-1} = \pi_{11} + \pi_{22} - 1$ and they then form a continuum in the states of the SSE. In the nonlinear model there is again a continuum of two-state Markov SSEs near the steady state and these exist provided the transition probabilities are sufficiently close to satisfying the resonant frequency condition. Within a neighborhood of the steady state this continuum is indexed by the deviation from the resonant frequency condition, with the states of the SSE uniquely determined once transition probabilities are given. Therefore, when properly interpreted, our results for an approximating linear model provide a satisfactory guide to the local results on E-stable SSEs in the corresponding nonlinear model.

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