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STOCHASTIC INTERTEMPORAL OPTIMIZATION IN DISCRETE TIME

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Abstract

The standard literature concerning intertemporal optimization in international finance is based upon certainty equivalence, and ignores risk and uncertainty. It therefore is not helpful concerning risk management and evaluation of the risk involved in the holding of international short-term debt. We solve a modification of the standard model of intertemporal optimization in discrete time, in an environment where the return to capital is stochastic. We impose the constraint that there be no default on the short-term debt. Thereby we derive benchmarks for optimal foreign debt, which will not be defaulted. We do not claim that the optimal debt is the same as the actual debt incurred. Witness the defaults and debt crises. Insofar as the actual debt exceeds the benchmark, the risk of default is increased. The main reasons for a deviation between the actual debt and the optimal debt is that the borrower is overly optimistic about the distribution function of the return to investment, and does not optimize subject to a "no default" constraint. We also consider an intertemporal optimization model involving extreme prudence. The lender, who may be an institutional investor, has infinite risk aversion and will only lend for projects where the profitability of the investment is almost sure. In this case also, we derive the optimal debt, which is our benchmark for risk management.

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Stochastic Intertemporal Optimization in Discrete Time

Wendell H. Fleming and Jerome L. Stein

1. The Need for a Paradigm of Risk Management of Short-term Foreign Currency Denominated Debt

Data on the credit rating of bonds issued in the first half of the 1990s suggest that investors in emerging market securities paid little attention to credit risk, or that they were comfortable with the high level of credit risk that they were incurring[1]. The compression of the interest rate yield spread prior to[2] and the subsequent turmoil in emerging markets have raised doubts about the ability of investors to appropriately assess and price risk.

Moody's indicated that there was a need for a "paradigm shift" that involves greater analytic emphasis on the risks associated with the reliance on short-term debt for otherwise creditworthy borrowers.

The literature in international finance concerning inter-temporal optimization in discrete time makes assumptions that implies to certainty equivalence[3]. These assumptions assume away the need for risk management, and fail to address the questions of how should one optimize under uncertainty, or evaluate what debt is likely to lead to default. We develop a paradigm for risk management: intertemporal optimization under uncertainty, with the constraint that there be no default on short- term foreign currency denominated debt in a finite horizon discrete time context.

2. A Discrete Time Finite Horizon Model, Risk and Risk Aversion

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[1] This section relies on International Monetary Fund, International Capital Markets, Washington DC (1999), and International Monetary Fund, Anticipating Balance of Payments Crises, Occasional Paper #186, (1999).
[2] The market expectations as embodied in interest rates did not widen significantly prior to the Mexican crisis. In the Asian crises, spreads hardly increased in the months prior to the floatation of the Bhat. The credit rating agencies and the market analysts all failed to signal the Asian crises in advance. They downgraded these countries only after the crises.
[3] See the reference to Obstfeld and Rogoff below. Similarly, the use of the Maximum Principle in continuous time assumes perfect certainty. Neither approach is useful in a world of risk and uncertainty. By contrast, Infante and Stein (1973) used dynamic programming to solve for intertemporal optimization in an environkment where there is not perfect knowledge. The derived suboptimal feedback control drives the economy to the unknown perfect certainty optimal path.

The contribution of our paper is as follows. In an earlier paper[4], we solved the problem of the optimal consumption, capital and foreign debt in continuous time over an infinite horizon, where the productivity of capital and the interest rate have Brownian motion components, which are correlated. Thereby we related the external shocks to the vulnerability of the banking sector. The technique of analysis is dynamic programming. Here, we analyze and solve a modification of the standard, well-known, model of intertemporal optimization. By considering this type of model, we achieve several objectives.

First: we show how to solve the intertemporal optimization problems without assuming away the risk by making assumptions that imply certainty equivalence, as is done in the literature[5]. Second by considering the standard two period version of model, the discrete period case can be solved by calculus, whereas the infinite horizon case discussed in our earlier paper employed the dynamic programming method with the technical mathematical difficulties encountered in the theory of continuous time stochastic control. Third: we do not assume that the disturbances are Brownian motion. The effects of different ways of describing the uncertainty, upon the optimal consumption, investment, the current account deficit and debt, are explicitly considered. Fourth: we focus upon the role of short- term debt, which must be repaid at the end of the second period. This is the "no bankruptcy" constraint. Fifth: The object is to select consumption and investment - and the resulting short term debt - in the first period to maximize the expected present value of consumption over both periods.

The constraint is that, regardless of the state of nature in the second period, there will be no default on the debt. Thereby we derive benchmarks for optimal foreign debt, which will not be defaulted. We do not claim that the optimal debt is the same as the actual debt incurred. Witness the defaults and debt crises. Insofar as the actual debt exceeds the benchmark, the risk of default is increased. The main reason for a deviation between the actual debt and the optimal

[4] Fleming and Stein, CESifo Working paper #204 (1999).

[5] The intertemporal optimization analysis in Obstfeld and Rogoff (1996: 60-87) makes assumptions that imply certainty equivalence. Hence risk is not considered in their resulting optimal consumption, investment and debt They are aware of this deficiency, and write (p.81) the following: "…consumption is determined according to the certainty equivalent principle. People make decisions under uncertainty by acting as if future stochastic variables were sure to turn out equal to their conditional means. Certainty equivalence is rarely a rational basis for decisions."

debt is that the borrower is overly optimistic about the distribution function of the return to investment.

Part 2.1 is a discussion of the structure of a modification of the standard model. The mathematical solution in part 2.2 shows how the intertemporal optimization model can be solved without making the assumptions that imply certainty equivalence. Part 3 concerns the optimization with finite risk aversion. A constraint is that there be no default.

There is no sure way to know what is the distribution of the stochastic variable, in our case it is the productivity of capital. Insofar as the resulting debt exceeds the derived optimal debt, the probability of default is increased. Part 4 considers an extremely prudent approach to intertemporal optimization by an agent who has infinite risk aversion. This would be the case if the lenders were institutional investors who are infinitely risk averse and will only lend for projects which are almost sure things. We derive optimal investment and debt in this most prudent case.

2.1 Modification of the Standard Model

Our horizon is a series of two period models with short-term borrowing, with initial and terminal conditions on debt and capital[6]. At the beginning of period t =1, capital K(1) > 0 and foreign debt L(1) = 0. The controls are consumption C(1) > 0 and investment I(1) $\geq$ 0 expenditures selected in period t =1. The excess of domestic investment over saving is the current account deficit, which is financed through short-term debt denoted L(2) that is due at the beginning of period t = 3. The debt is denominated in US dollars; hence C(1), I(1) and GDP denoted by Y(1) are also measured in US dollars. The constraints are that: (i) Consumption is positive. (ii) There is no default. At the beginning of the third period the dollar denominated debt plus interest must be repaid. Our debt constraint is that: L(3) = L(1) = 0. That is why we refer to the debt as "short-term" debt.

A two period model also requires a constraint on the capital at the beginning of period three: K(3). We could require that K(3) be a multiple of the initial K(1). Here we select the

[6] Otherwise, the Fleming-Stein (1999) infinite horizon model should be used. But in that case, there is no short- term debt: the debt need never be repaid, but must always be serviced.

multiple equal to unity, so that K(3) = K(1). This means that (iii) the initial capital must be returned: K(3) = K(1), which implies that I(1) + I(2) = 0, so that the process of short-term debt financing can be repeated in the third period[7].

The uncertainty concerns the productivity of capital and investment, which determine the dollar value of GDP equal to net value added, denoted Y(2), and consumption C(2) in the second period. The productivity of capital and investment is b(2), discussed below.

The criterion function is equation (1): the maximization, over the set of controls and the constraints (i)-(iii) above, of the expectation [E] of the present value [J] of the utility of consumption. The discount factor[8] β is the ratio of future/present utility. The expectation is discussed below. There is risk aversion: finite or infinite. The utility function at each period is HARA, $U(C(t)) = (1/\gamma)C^{\gamma}$, where positive $(1-\gamma) > 0$ is risk aversion. This is a concave positively sloped function, with an infinite slope at a zero level of consumption. When $\gamma < 0$, utility has an upper bound at zero. Three cases are considered. The first two assume that $(1-\gamma) > 0$ is finite. The case where $\gamma = 0$ implies that $U(C(t)) = \ln C(t)$. The third case, discussed in part 4, is very important and less well known. It involves a very conservative approach to risk management. It is called the Large Deviations [LD] approach to risk. There are three parts to the LD approach. First: is that there is infinite risk aversion, $\gamma \Rightarrow -\infty$, or $(1-\gamma) \Rightarrow \infty$. Second, the probability of the bad case (1-p) is negligible: that is the good case is almost a "sure thing". This is modeled as $(1-p) = e^{-\alpha(1-\gamma)}$, $\alpha > 0$. Third $(1-p)^{1/1-\gamma} = e^{-\alpha} = B$, where $1 > B > 0$. The meaning of the very conservative LD approach is discussed below.

(1) $E[J] = \text{Max}\ \{U(C(1)) + \beta\, E_{b(2)}[U(C(2))]\}$, $C(1) > 0$, $I(1) \geq 0$.

The dollar denominated debt L(2) that exists at the beginning of period t=2, equation (2), is the current account deficit in period t=1, equal to consumption C(1) plus investment I(1) less the GDP denoted Y(1).

(2) $L(2) = C(1) + I(1) - Y(1)$.

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[7] It is quite easy to use the constraints: L(1) = a, L(3) = b, K(1) = c > 0 and K(3) = eK(1), where e > 0. We have selected a = b = 0, and e = 1.

[8] In our model the two stages are viewed as part of a cycle, which repeats forever. If future cycles are given weight equal to the initial cycle, it makes most sense to think of discount factor $\beta = 1$.

Initial capital is K(1) > 0, and investment I(1) $\geq$ 0. The capital at the beginning of period t=2 denoted K(2) is equation (3). The investment in period t=2 is I(2) = -I(1), since we require that K(3) = K(1), to restart the process of short term borrowing.

(3) K(2) = K(1) + I(1).

The uncertainty concerns consumption in period two C(2), described by equation (4). Consumption, debt and capital are measured in real $US. It is the GDP plus capital, Y(2) + K(2), less the repayment of the debt plus interest sL(2) = (1+r)L(2) less the return of the initial capital K(1), to restart the process.

(4) C(2) = Y(2) + K(2)- (1+r)L(2) - K(1)

The uncertainty involves the GDP in period t=2, denoted Y(2). The production function is Y(t) = b(t)K(t): there is a fixed proportion between GDP and capital K(t). The limiting factor is always capital[9]. The productivity of capital or investment b(2) = Y(2)/K(2) in period t=2 is unknown when the investment decision is made at t=1. The productivity of investment b(2) is <u>stochastic</u> for the following reason. Dollars are borrowed at interest rate r to purchase capital and produce an output, which is sold in the world market. The dollar value of the output depends upon several factors: the terms of trade (export/import prices), the real exchange rate of the country and the productivity of the investment, measured in domestic currency. If the terms of trade deteriorate, the investment is ill advised or the real exchange rate depreciates, the productivity of the investment b(2) declines. Then the repayment of the dollar denominated debt is more costly. Instead of viewing the effect of exchange rate uncertainty upon the interest payments denominated in foreign currency, we view everything via the productivity of investment[10].

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[9] The production function is: Y(t) = min[b(t)K(t), a(t)W(t)] = b(t)K(t), where W= other inputs.

[10] In this paper, the short term rate of interest is known, but the productivity of capital b(2) is unknown. A good example of the uncertainty, and our use of b(2) to describe it, concerns Mexico and Thailand. The firms and the banks borrowed US dollars on the assumption that the exchange rate would continue to be fixed relative to the US dollar. When the terms of trade declined, the return in domestic currency declined. The firms had difficulty in repaying the banks that, in turn, had more non-performing loans. It then was difficult for the banks to repay the foreign creditors, and the exchange rate depreciated. The depreciation aggravated the decline in b(2) arising from the decline in the terms of trade. With a depreciated currency, the net value added of the economy commanded fewer dollars to repay the loans.

The uncertainty is described by the net return on investment is b(2) - r, the productivity of capital less the interest rate. The range of b(2) is $r \pm a/2$, $a > 0$. The values of the net return b(2) - r are symmetrical around zero with a range $a > 0$, as described below, with probabilities (p, 1-p), $1 > p > 0$, in the good and bad case respectively. This is not Brownian motion, but a simple and general formulation that makes minimal assumptions about the distribution function.

b(2)	Pr(b)	
$b^+(2) = r + a/2$	$1 > p > 0$	good case
$b^-(2) = r - a/2$	$(1-p) > 0$	bad case

Expected net return $E[b(2)-r] = a(p - 1/2)$; range $[b(2)-r] = a > 0$

Using the production function and nature of the uncertainty, we write the consumption in period t=2 as equation (4.1), where $Y(1) = b(1)K(1)$.

(4.1) $C(2) = s[b(1)K(1) - C(1)] + [b(2) - r]I(1) + b(2)K(1)$

Consumption in period 2 has three components. The first term is the interest plus principal on saving $s[b(1)K(1) - C(1)]$, which can be loaned or borrowed at rate r. It is not stochastic and is known at t = 1. The second term is the net return on investment I(1), the productivity of capital b(2) less the interest rate r. The productivity of capital b(2) less the interest rate is the stochastic variable that is unknown when investment decisions are made in period t=1. The third term is the consumption C(2) that would be possible if there is neither saving nor investment in period t=1. It is the initial capital K(1) times its productivity b(2), where b(2) is unknown at t=1.

2.2 Mathematical Technique and Solution

We solve our modification of the standard model by taking explicit account of the uncertainty, rather than by using the certainty-equivalence approach in the literature, and by explicitly using the "no bankruptcy" constraint . Consumption C(2) is a stochastic variable. When the productivity of capital takes on the good value $b^+(2) = r + a/2$, with probability p, then consumption $C^+(2)$ is equation (5); and when the productivity of capital takes on the bad value $b^-(2) = r - a/2$, with probability (1-p), consumption is $C^-(2)$ in equation (6).

(5) $C^+(2) = (1+r)[b(1)K(1) - C(1)] + (a/2)I(1) + (r + a/2)K(1)$

(6) $C^-(2) = (1+r)[b(1)K(1) - C(1)] - (a/2)I(1) + (r - a/2)K(1)$

If there is optimal risk management in period t=1, the country would select the controls $C(1) > 0$, $I(1) \geq 0$ to maximize the expectation of the present value of utility of consumption, E[J], which is strictly concave, on the four-sided convex polygon **G** defined by the controls and constraints: $C(1) > 0$, $C^+(2) > 0$, $C^-(2) > 0$, $I(1) \geq 0$. There is no default: $L(3) = 0$, the debt plus interest must be repaid. The capital inherited K(1) must be repaid at the beginning of the third period: $K(3) = K(1)$, so that the process can be repeated[11].

(7) $\max E[J] = \mathrm{MAX}_{\Gamma} \{ (1/\gamma)C^{\gamma}(1) + \beta(1/\gamma)[p\, C^+(2)^{\gamma} + (1-p)C^-(2)^{\gamma}]\}$,

over the controls and constraints **G**. The crucial partial derivatives, to be used in the solution, are equations (5.1) - (5.2) in the good case, and (6.1)-(6.2) in the bad case.

Probability $p > 0$	Probability $(1-p) > 0$
(5.1) $dC^+(2)/dC(1) = -s = -(1+r)$;	(6.1) $dC^-(2)/dC(1 = -s = -(1+r)$
(5.2) $dC^+(2)/dI(1) = + a/2$;	(6.2) $dC^-(2)/dI(1) = - a/2$

Since E[J] is strictly concave over **G**, the maximum is at a unique $C^*(1)$, $I^*(1)$, which is either interior to Γ or on the boundary $I(1) = 0$.When the maximization is interior to Γ, it is found by setting the partial derivatives of E[J] with respect to C(1) and I(1) equal to zero. This gives equations (8) and (9):

maximization

(8) $C^{\gamma-1}(1) = \beta s\{p[C^+(2)]^{\gamma-1} + (1-p)[C^-(2)]^{\gamma-1}\}$

(9) $\beta\{ p[C^+(2)]^{\gamma-1} (a/2) - (1-p)[C^-(2)]^{\gamma-1} (a/2)\} = 0$

Variables $C^+(2)$ and $C^-(2)$ are defined in (5) and (6), respectively. We solve (8) and (9) via a transformation (10.1)-(10.2). The consumption in the second period is a proportion of that in the first period.

(10.1) $[C^+(2)]^{\gamma-1} = A^+ [C(1)]^{\gamma-1}$

(10.2) $[C^-(2)]^{\gamma-1} = A^- [C(1)]^{\gamma-1}$

[11] See the earlier footnote how we can also use the assumption that $K(3) = eK(1) > 0$, without specifying that $e=1$.

We determine the values of A^+ and A^- by substituting (10.1) and (10.2) into the maximizing relations (8) and (9), and obtain (11) and (12). The latter two equations concern the maximization with respect to control variables C(1) > 0 and I(1) > 0, and we solve for the values of A^+ and A^-.

(11) $pA^+ + (1-p)A^- = 1/\beta s$

(12) $pA^+ - (1-p)A^- = 0.$

Consider the case where I(1) > 0, and equation (12) is an equality. The resulting values of A^+ and A^- are equations (13.1)-(13.2).

(13.1) $A^+ = 1/2p\beta s$

(13.2) $A^- = 1/2(1-p)\beta s$

Using (13.1) and (13.2) in (10.1) and (10.2), we obtain the values of consumption C(2) in period two relative to the optimal control C(1), equations (16.1) and (16.2).

(14.1) $C^+(2) = (2p\beta s)^{1/1-\gamma}\, C(1)$

(14.2) $C^-(2) = (2(1-p)\beta s)^{1/1-\gamma}\, C(1)$

Now, we can solve for the optimal controls C(1) and I(1). Substitute equation (14.1), the consumption that results in the good case, in equation (5) to obtain equation (15). Similarly substitute equation (14.2), the consumption that results in the bad case, in equation (6), to obtain equation (16). These two equations permit us to solve for the optimal controls, c(1) = C(1)/K(1) and i(1) = I(1)/K(1), as a fraction of the initial capital K(1), when there is uncertainty about the future productivity of capital. These equations are graphed in figure 1. The crucial parameters S_1, S_2, and N are defined in table 1 below.

(15) $S_1\, c(1) – (a/2)\, i(1) = N$

(16) $S_2\, c(1) + (a/2)\, i(1) = N - a$

The condition for an interior maximum is that the solution to (15), (16) satisfy c(1) > 0, i(1) > 0. This is equivalent to $S_1(N-a) > S_2N$. See formula (18) below. When $S_1(N-a) \leq S_2N$, the maximum occurs on the boundary i(1) = 0.

These two equations are quite different from the equations in the literature which assume certainty equivalence, because we explicitly consider the nature of the uncertainty. The maximization procedure gives us the relation between c(1) and i(1) in the good case (equation

15), and in the bad case (equation 16). In each case, there is a constraint that there is no default. In the good case, the net return on investment is positive; so both c(1) and i(1) can increase without adversely affecting C(2). This implies the positively sloped curve, equation (15). In the bad case, the net return is negative. Higher investment i(1) must lower consumption C(2), unless c(1) is lower, which produces the negatively sloped curve equation 16. The controls c(1) and i(1) in the first period must be chosen before we know whether there will be a good or a bad outcome. Consequently, the optimal controls are given by the intersection of the curves of equations (15)-(16). Optimal consumption/capital c(1) = C(1)/K(1) is equation (17). Optimal investment/capital i(1) = I(1)/K(1) is equation (18). Optimal debt/capital f(2) = L(2)/K(1) is equation (19). These are general results.

There are three cases, based upon the attitude towards risk, which determine the basic parameters in table 1. In part 3, we consider the case of general risk aversion ,where $\gamma < 1$. The specific special logarithmic case $\gamma = 0$ produces a simple solution, discussed in part 3. The large deviations [LD] case, discussed in part 4, is an extremely conservative attitude to risk. There is infinite risk aversion $\gamma \Rightarrow -\infty$, but the only investments that are undertaken are those where the probability of the bad event, is negligible.

3. Risk Management: Risk and Finite risk aversion

Table 1 and equations (17)-(19) contain our solution to the risk management in the discrete time model where there is a short-term debt that must be repaid at the beginning of the third period. In this part, we discuss the case of finite risk aversion. In part 4, we discuss the "large deviations" [LD] approach, where there is infinite risk aversion. The optimal foreign debt f(2) = L(2)/K(1) incurred during the first period, equation (19), is simply the trade deficit. It is equal to optimal consumption plus optimal investment less GDP, which is equal to optimal investment less optimal saving all per unit of capital: f(2) = i(1) - [b_1 - c(1)]. This difference is equal to optimal new borrowing. We examine the various components; investment i(1) and saving h(1) =[b(1) - c(1)] .

3.1 Optimal Consumption and Saving

Optimal consumption/capital $c(1) = C(1)/K(1)$ is equation (17), table 1, in the general case. The value of the numerator (2N - a) is the same in all three cases, but the value of the denominator $(S_1 + S_2)$ is based upon the particular cases in table 1. In the text, we explicitly consider two cases: the $\gamma = 0$ here and the LD case in part 4.

The numerator in equation (17) can be expressed in familiar terms. Suppose that current capital and its income $K(1) + Y(1) = (1 + b(1))K(1)$ were loaned out at the known short term interest rate r. The assumption that $K(3) = K(1)$ means that K(1) must be returned, that is $I(2) = -I(1)$. With certainty, the resources available in period t=2 would be $[s(1 + b(1))- 1]K(1)$. Hence the present value of the net resources per unit of capital K(1) that would be available from a riskless position is Y*, defined in equation (20). This quantity is known with certainty. Refer to Y* as "safe wealth". Therefore the optimal consumption/capital equation (17) can be written as (17a).

(20) $Y^* = [s(1 + b(1))- 1]/s$ "SAFE WEALTH".

(17a) $c(1) = 2sY^*/(S_1 + S_2)$

The denominator, as is seen in the columns of table 1, varies in the three cases. In the first column where $1 > \gamma > \infty$, but $\gamma \neq 0$, the denominator $(S_1 + S_2)$ involves uncertainty or risk. It is positively related to the probability of the good event for $p > 1/2$. As the expected net return rises, the denominator rises, and optimal consumption/capital declines.

In the $\gamma = 0$ case, the optimal consumption/capital is equation (17b). It does not involve uncertainty. It is a multiple of "safe wealth". If the future utility is given the same weight as present utility, then optimal consumption is 1/2 of "safe wealth".

(17b) $c(1) = Y^*/(1+\beta)$.

Optimal saving/capital is equation (21), in the case where $\gamma = 0$. It is equal to $Y(1)/K(1) = b_1$ less consumption/capital from equation (17b).

(21) $h(1) = Y(1)/K(1) - C(1)/K(1) = b(1) - Y^*/(1+\beta)$.

This equation does involve inter-temporal consumption smoothing. Saving will be positive if the current output/capital b(1) exceeds the discounted value of "safe wealth" $Y^*/(1+\beta)$. All of these magnitudes are known with certainty at the time decisions are made. The known current productivity of capital $b(1) = Y(1)/K(1)$, the known rate of interest r, are measurable and the relative weight on future utility β is a value judgment.

3.2 The Investment Function

In the standard literature, the stock of capital K(t+1) is selected such that the expected marginal productivity is equal to the exogenous interest rate. This is certainty equivalence. If the expected marginal productivity of capital is constant and exceeds the interest rate, then an infinite rate of investment and debt are optimal. In our two period analysis of optimal short-term debt, the productivity of capital $b(t) = Y(t)/K(t)$ is independent of the total stock of capital. Unlike the certainty equivalence case, our equation for optimal investment/capital i(1) given by equation (18), implies very different conclusions. Equation (18) and table 1 summarize our results in three cases. In this part of the text, we discuss the case: [B] where $\gamma = 0$, and part 4 discusses the [LD] large deviations case.

Equation (18) for investment/capital $i(1) = I(1)/K(1)$ is expressed as equation (18a) in the case where $\gamma = 0$. We use the definitions in table 1. The expected net return $x = E[b(2)-r] = a(p-1/2)$ is a crucial variable. The indicator function $\chi = 1$ when $x > \rho$ and $\chi = 0$ otherwise[12]. Term ρ defined in (18b) corresponds to a risk premium. It is positively related to the downside risk that $b(2) = r - a/2$, and negatively related to safe wealth.

(18a) $i(1) = \chi[\beta/(1+\beta)]sY^* (x - \rho)$.

(18b) $\rho = (a/2)^2 \ / [\beta/(1+\beta)]sY^* > 0$.

This equation for optimal investment/capital is plotted in figure 2 as the broken line i(1). It is $i(1) = 0$ for $x \leq \rho$, and linear in $x > \rho$. In the standard literature, which relies upon certainty equivalence, if the expected net return - equal to the expected marginal product of capital less the interest rate - is positive, investment is positive. If the marginal product of capital were relatively constant, then investment would be unbounded. Our "intertertemporal optimization" approach equation (18a), on the other hand, is very sensitive to risk. There will be no

[12] This is the condition that indicator function: $\chi = 1$ when $[(N-a)S_1 - NS_2\ ] > 0$, and $\chi = 0$ otherwise..

investment unless the expected net return E[b(2) - r] exceeds the risk premium ρ defined in (18b).

3.3. Optimal foreign debt/capital

We are now able to answer the question of risk management. How should we evaluate the risk involved with holding or issuing short-term debt? The optimal foreign debt per unit of capital f(2) = L(2)/K(1) incurred during the first period is simply the trade deficit. It is equal to optimal investment i(1) in equation 18) less optimal saving h(1) in equation 21, all per unit of capital. In the case where $\gamma = 0$, it is equation (22), graphed in figure 2.

(22) $f(2) = i(1) - h(1) = \chi[\beta/(1+\beta)]sY^* (x - \rho) - [b_1 - Y^*/(1+\beta)]$.

The optimal foreign debt f(2) is the difference between i(1) and h(1). Investment should only be positive if the expected net return x exceeds the risk premium ρ. Saving h(1) is positive. In the case where $\gamma = 0$, saving is independent of the expected return or risk factors.

The country should incur short-term debt if the expected net return exceeds quantity $0B > \rho > 0$, and should be a short- term lender if the expected net return is less than 0B. A short-term foreign debt is excessive, the economy is vulnerable, if the foreign debt exceeds the f(2) line in figure 2. All of the quantities in equation (21) are theoretically measurable, except for two preference or value judgment factors: the discount factor equal to the relative weight on future utility $\beta/(1+\beta)$ and risk aversion $(1-\gamma)$.

TABLE 1. Definitions of Crucial Terms: three cases

	CASE [A] $\infty < \gamma < 1$	CASE [B] $\gamma = 0$	CASE [LD] Large Deviations $(1- p) = e^{-\alpha(1-\gamma)}$, $\alpha > 0$, γ=>-∞, $(1-p)^{1/1-\gamma} = e^{-\alpha} = B$ 1>B> 0 weight on bad case
S_1 weight on good case	$(2ps\beta)^{1/1-\gamma} + s$	$s(2p\beta + 1)$	$1 + s$
S_2 weight on bad case	$(2(1-p)s\beta)^{1/1-\gamma} + s$	$s(2(1-p)\beta+1)$	$B + s$
$S_1 - S_2$	$(2ps\beta)^{1/1-\gamma} - (2(1-p)s\beta)^{1/1-\gamma}$	$2s\beta(2p - 1)$	$1 - B > 0$
$S_1 + S_2$	$2s + (2s\beta)^{1/1-\gamma} [p^{1/1-\gamma} + (1-p)^{1/1-\gamma}]$	$2s(1+\beta)$	$(1+B +2 s)$
S_1/S_2		$(2p\beta + 1)/(2(1-p)\beta+1)$	$(1+s)/(B+s) > 1$
N	$(1+ b(1))s + (a/2) - 1 = sY^* + (a/2)$	$sY^* + (a/2)$	$sY^* + (a/2)$
(N – a)	$(1+ b(1))s - (a/2) - 1 = sY^* - (a/2)$	$sY^* - (a/2)$	$sY^* - (a/2)$

Note: The values of the net return b(2) - r are symmetrical around zero with a range a > 0, and probabilities (p, 1-p). Expected net return $x = E[b(2)-r] = a(p - 1/2)$. The interest rate is r and define s = 1 + r. "Safe wealth" is defined as $Y^*= [(1+ b(1))s - 1]/s$, and the risk premium in case [B] is defined as $\rho = (a/2)^2/[\beta/(1+\beta)sY^*]$.

Table 2

Optimal controls: consumption/capital, c(1) saving/capital h(1), investment/capital i(1), and the debt/capital f(2).

(17) $c(1) = (2N - a)/ (S_1 + S2)$

(21) $h(1) = b(1) - c(1)$

(18) $i(1) = \chi(a/2)[(N-a)S_1 - NS_2]/[(S_1+S_2)]$

(19) $f(2) = [C(1) + I(1) - Y(1)]/K(1) = i(1) - h(1) = c(1) + i(1) - b(1)$

See table 1 for definitions of N, S_1, S_2.

Indicator function: $\chi = 1$ when $[(N-a)S_1 - NS_2] > 0$, and $\chi = 0$ otherwise.

4. Extreme Prudence: Large Deviations Model

The optimization has focused upon a borrower facing a given rate of interest. The decision, concerning optimal investment in period t=1 and the optimal debt carried into period t=2, is constrained to satisfy the condition that there be no default, $L(3) = 0$, regardless of whether the good or bad outcomes occur. Borrowers with low risk aversion select higher investment and debt ratios than what are selected by those who are more risk averse. This follows from the inference that investment i(1) in equation (18) is directly related to S_1/S_2 . According to the values in column [A] of table 1, when $p > 1/2$ ratio S_1/S_2 is directly related to coefficient γ.

If the lenders are institutional investors that manage pension funds, they may be infinitely risk averse. The distribution of the net return is subjective, and no one can objectively justify what is the probability of a successful outcome. On the basis of sad exerience, the lenders may not share the optimism of borrowers that default will not occur[13]. Institutional investors may only be willing to lend for investments where the probability is minimal that the productivity of capital is below r - the interest rate on short term debt. We develop an alternative approach to optimization, called the Large Deviations [LD] model. The lenders may only be willing to lend at short term rate r if the borrower optimizes according to the lenders' criteria.

We take the viewpoint of the conservative lender who wants to manage risk, and derive what these lenders consider to be optimal investment, consumption and debt. Although there is infinite risk aversion $(1-\gamma) \Rightarrow \infty$, the only investments that are financed at rate r are those where the probability of a successful outcome is almost unity, $p \sim 1$. This is a very conservative investment strategy: the LD case. This gives us a benchmark for vulnerability or for risk management. If the debt exceeds the optimal debt in the LD case, then the lender considers that the economy is vulnerable to shocks, and the borrower is likely to default.

4.1 Optimal investment

[13] See the introductory part of this paper.

There are three main features of the LD model. First: There is infinite risk aversion, the good case is almost a sure thing, and it is most unlikely that the bad event will occur. The probability of the good event $p \sim 1$, and the probability of the bad event
$(1-p) = e^{-\alpha(1-\gamma)}$ where risk aversion $(1-\gamma) \Rightarrow \infty$ and $\alpha > 0$. Second: The values of the net return $[b(2) - r]$ are either $(a/2) > 0$ or $-(a/2) < 0$. Since $p \sim 1$, the expected net return $E[b(2) - r] = (a/2) > 0$. Therefore the expected net return is equal to the symmetrical upside and downside risk, albeit with different probabilities. Define $x = E[b(2) - r] = (a/2)$, the expected net return equal to the downside risk. Third: The crucial parameter $B = (1-p)^{1/1-\gamma} = e^{-\alpha}$. Since $\alpha > 0$, then $1 > B > 0$. We refer to B as the weight placed upon the bad event. It combines infinite risk aversion with minimal risk.

The solution for the optimal value of investment is given by substituting in equations (18), the values of $S_1 = (1+s)$ and $S_2 = (B + s)$ in column 3 of table 1. A condition that investment be positive, that the indicator function $\chi = 1$, is that inequality (23a) is satisfied[14].

(23a) $\chi=1$ if $sY^*(1-B)/(1+2s+B) > a/2$. Otherwise $\chi = 0$.

Denote $\rho^* = sY^*(1-B)/(1+2s+B)$, and $x = a/2 = E[b(2)-r]$. The investment/capital in the Large Deviations case is equation (23).

(23) $i(1) = \chi[x(\rho^* - x)]$

Investment will be positive if inequality (23b) is satisified.

(23b) $\rho^* > x > 0$.

Figure 3 plots investment $i(1) = I(1)/K(1)$ against x which, in the LD case, is both the expected net return $E[b(2)-r]$ and the downside risk a/2. Optimal investment is a parabola, with $i(1) = 0$ at $x = 0$ and $x = \rho^*$. It reaches a maximum at $\rho^*/2$. The logic of the parabola is that when $x = E[b(2)-r] = 0$, there is no net return so it does not pay to invest in risky assets. As x rises, the expected net return rises and investment is induced. Since $x = a/2$ is equal to the downside risk, as x rises above $\rho^*/2$, the risk element dominates and decreases investment. At $x = \rho^*$, the risk has total domination and optimal investment returns to zero.

[14] This is the condition $[(N-a)S_1-NS_2] > 0$ for positive investment.

Two investment functions are plotted in figure 3, for a high and a low value of B. As parameter B the weight on the bad event rises, the value of ρ^* declines. The parabola of investment/capital declines, with a smaller range of x for which there is positive investment.

4.2 Optimal Debt.

Optimal debt per unit of initial capital f(2) is optimal investment less optimal saving. Optimal investment i(1) is equation (23) the parabola; and optimal saving h(1) is equation (24) the straight line. Saving h(1), equation (24), is independent of the expected net return x. It is positively related to the weight B placed upon the bad event and to the current productivity of capital b(1), and is negatively related to safe wealth Y*. As the value of B rises, the saving function graphed in figure 3 rises.

(24) $h(1) = [b(1) - 2sY^*/(1+B+2s)]$

The equation for optimal debt/capital f(2) is equation (25). It is the vertical distance between the two curves in figure 3. There will be foreign debt for $x_2 > x > x_1$.

(25) $f(2) = \chi[x(\rho^* - x)] - [b(1) - 2sY^*/(1+B+2s)]$,

where $x = E[b(2) - r] = (a/2)$ and $\rho^* = sY^*(1-B)/(1+2s+B)$

The crucial variable here is fraction $B = (1-p)^{1/1-\gamma} = e^{-\alpha}$, the weight on the bad event. This is a quantity that the lender/borrower must select. Weight B affects both saving and investment. As B rises to B', the saving function rises from h(1) to h'(1). The rise in B lowers ρ^*, and the investment function declines from i(1) to i'(1) in figure 3. For any level of x = E[b(2)-r] = a/2, the optimal foreign debt declines. There will be foreign debt for $x_4 > x > x_3$.

5. Conclusion

The standard model of intertemporal optimization is based upon certainty equivalence and ignores risk and uncertainty. It therefore contains no useful information concerning risk management and evaluation of international short-term debt. We solve a modification of the standard model of intertemporal optimization in an environment where the return to capital is stochastic, and we impose the constraint that there be no default on the short-term debt. Thereby, we derive benchmarks for optimal foreign debt in a world of uncertainty. Insofar as the actual debt exceeds the benchmark, the risk of default is increased. The main reasons for a deviation between the actual debt and the optimal debt are that the borrower is overly optimistic about the distribution function of the return to investment, and does not optimize with the "no default" constraint.

We also consider an intertemporal optimization model involving extreme prudence. The lender, whom we think of as an institutional investor, has infinite risk aversion and will only lend for projects where the profitability of the investment is almost sure. In this case also, we derive the optimal debt, which is our benchmark for risk management.

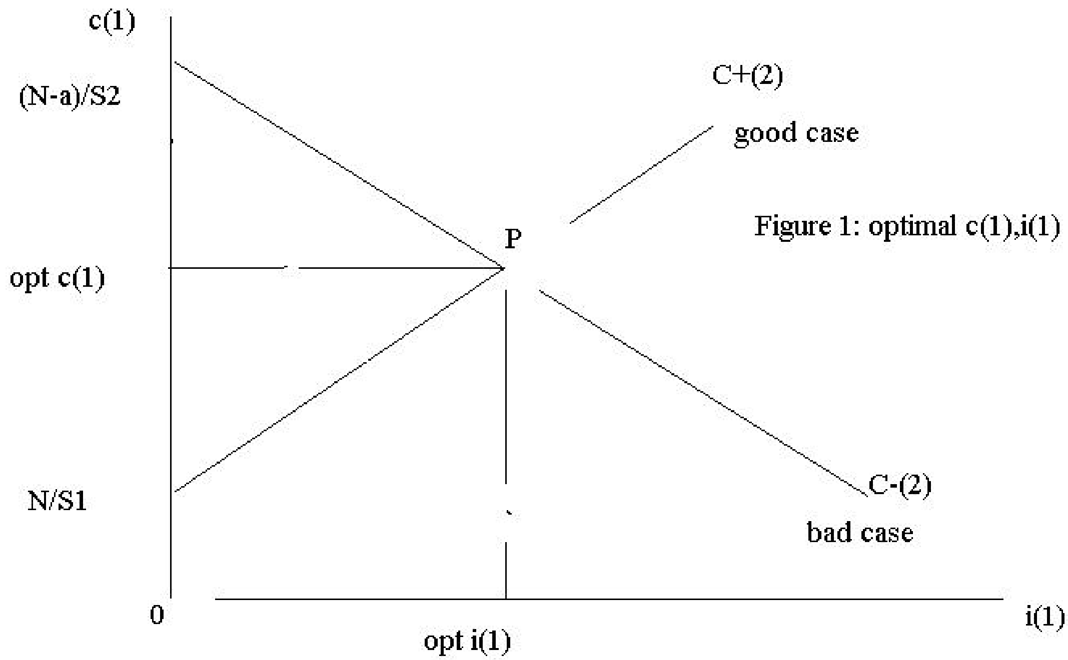
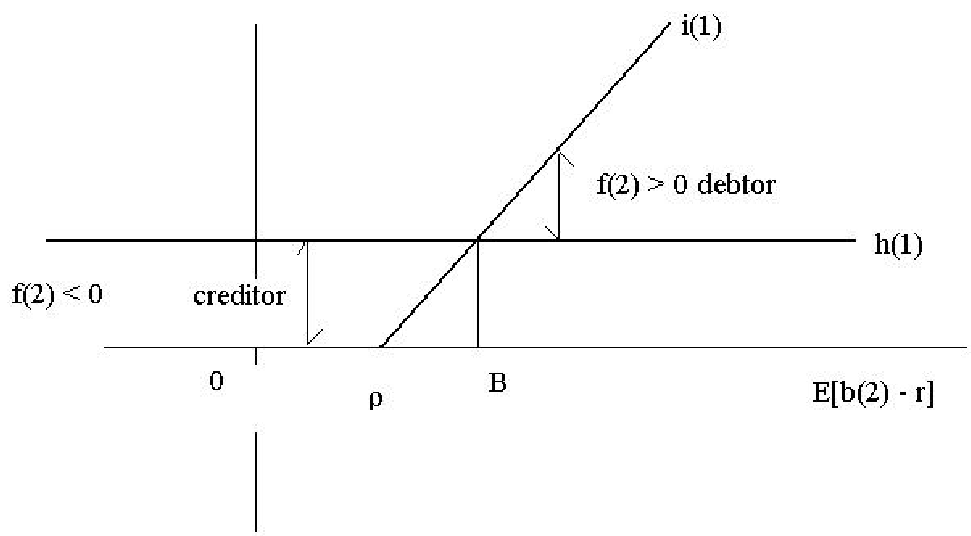

c(1)
(N-a)/S2
C+(2)
good case
Figure 1: optimal c(1),i(1)
P
opt c(1)
N/S1
C-(2)
bad case
0
i(1)
opt i(1)

Figure 2



f(2) = i(1) - h(1); f(2) > 0 debtor; f(2) < 0 creditor

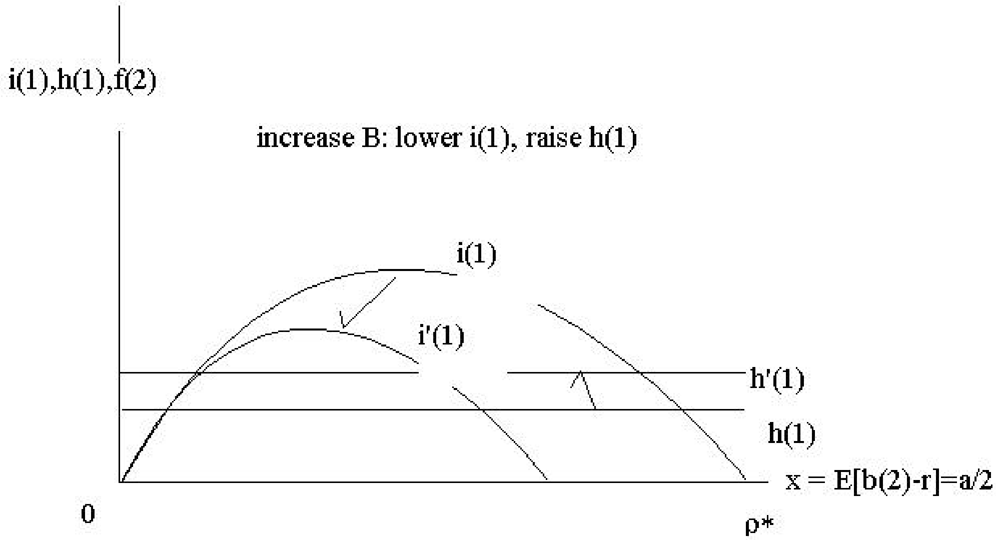


Figure 3. Large deviations. Investment i(1), saving h(1)
debt f(2) = i(1) - h(1)

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