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SADDLES, INDETERMINACY AND BIFURCATIONS IN AN OVERLAPPING GENERATIONS ECONOMY WITH A RENEWABLE RESOURCE

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Abstract

We incorporate a renewable resource into an overlapping generations model with standard, well-behaved utility and constant returns to scale production functions. Besides being a factor of production the resource serves as a store of value. We characterize dynamics, efficiency and stability of steady state equilibria and show that the nature of steady state equilibrium depends on the value of the intertemporal elasticity of substitution in consumption. In particular, if that elasticity is at least half, but not exactly one, stationary equilibria are saddle points. The stationary equilibrium is stable when the intertemporal elasticity of substitution is unity. For smaller values of that elasticity we use a parametric example to demonstrate the existence of stable equilibria (indeterminacy) and a subcritical flip bifurcation. Hence, an overlapping generations economy with a renewable resource can display cycles and indeterminacy even in the absence of externalities or imperfect competition.

Keywords: Overlapping generations, renewable resources, bifurcations

JEL Classification: D90, Q20, C62

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1. Introduction

The stability properties of overlapping generations models have been subject to a fairly large amount of research since the mid 1980's. It has been shown how idealized business cycles may appear in a purely endogenous fashion even though "fundamentals" of the system, i.e., tastes, endowments and technologies or economic policies, do not vary over time. Endogenous business cycles have been known to be possible in overlapping generations models since Gale (1973). To mention a few more recent examples, Farmer (1986) and Reichlin (1986) have shown using slightly different models the existence of limit cycles (Hopf bifurcations) in planar systems, especially in the one-sector overlapping generations model of capital accumulation. By applying the theory of flip bifurcations Grandmont (1985) has shown how in a particular version of this class of models periodic equilibria can occur. Grandmont (1998) presents an intuitive survey of some recent developments, which have utilized geometric methods. For a comprehensive survey of the field, the reader may consult Azariadis (1993).

Another issue associated with the properties of dynamic systems is indeterminacy. It has been shown more recently that, for instance, a one-sector real business cycle model with sufficient aggregate increasing returns to scale or a multisector model that has constant returns to scale and market imperfections, may exhibit indeterminate steady state (i.e. sink) that can be exploited to generate business cycles driven by "animal spirits".[1] Benhabib and Farmer (1999) provide a survey of this literature from the macroeconomics viewpoint.

To demonstrate either bifurcation or indeterminacy in an overlapping generations model, or in a real business cycle model, one usually has to make quite specific assumptions about the fundamentals, e.g., postulating either increasing returns to scale or externalities.

These stability and indeterminacy issues have not been studied carefully in models with renewable resource use, like forestry or fisheries. Traditional theories of renewable resource use assume an infinitely lived agent or a social planner, and demonstrate that there is one steady state equilibrium, which is a saddle. Equilibrium is a function of resource price and exogenous real interest rate (for economics of forestry and fisheries, see e.g. Clark

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[1] Also the terms "sunspots" and "self-fulfilling beliefs" are used interchangeably in the literature to refer to the same phenomenon.

1990 and Johansson and Löfgren 1985). These models do not account for the fact that in practice renewable resources are important stores of value between different generations.[2] Hence, one can ask whether this standard renewable resource analysis is robust in an overlapping generations economy, where agents have a finite life but resource stock may grow forever, and where the real interest rate is endogenously determined.

Recent studies (Kemp and Long 1979, Löfgren 1991, and Mourmouras 1991, 1993) focusing on the sustainable use of renewable resources within the overlapping generations framework have established the generally well-known fact that competitive equilibria in overlapping generations economies may be inefficient.[3] These papers share the common feature that they study the steady state equilibrium without analyzing its transition dynamics and thereby the stability properties. This is an unfortunate drawback for several reasons. First, it is not obvious what the dynamic properties are when the model includes a renewable resource with its own dynamics. Second, one may argue that stability properties of the renewable resource exploitation are important especially for policy. If the utilization of the resource tends to be unstable, competition may more easily lead to the destruction of the whole resource, which naturally necessitates a more careful resource management.[4] Olson and Knapp (1997) is an interesting study of an overlapping generations economy with an exhaustible resource. With the exception of the resource type, their basic model is quite similar to ours.

Our purpose is to examine the dynamic properties of a conventional overlapping generations economy augmented with a renewable resource which serves both as a factor of production and a store of value. Because a renewable resource has its own dynamics and growth function, we will get a planar system with harvesting and the resource stock as dynamic variables. We characterize the steady state equilibrium of this overlapping

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[2] Tobin (1980), for instance, when criticizing the role of money as a store of value in overlapping generations models, pointed out that “land and durable goods, or claims upon them are principal stores of value” (p. 83).

[3] Kemp and Long (1979) demonstrate that a competitive economy with constant population may under-harvest its renewable resources as a consequence of the resource being inessential for production. In a different vein, Mourmouras (1993) shows that both a low rate of resource regeneration relative to population growth and a low level of saving may lead to unsustainable use of renewable resources, so that consumption declines over time.

[4] In addition to the above references, see e.g. Amacher et al (1999) for an analysis of the effects of forest and inheritance taxation on harvesting stand investment and timber bequests in an OLG model with one-sided altruism.

generations economy, compare competitive and efficient solutions, and in particular, study its stability properties, which have thus far remained unexplored in the literature.

We construct a general equilibrium overlapping generations model where agents live two periods and there is no population growth. The young are endowed with one unit of labor and earn a competitive wage. They can consume or save in the financial asset or buy the available stock of the renewable resource from the firm. During the first period of their lives the young inelastically supply labor to firms, which transforms labor and resource, which they buy from the old, into output by constant returns to scale technology. As the focus is entirely on the extractive use of resource, we omit amenity services provided by the resource. The resource stock may be interpreted as either forests or fisheries (with well-defined property rights over fish stocks). Unlike Kemp and Long (1979) and Mourmouras (1993), who make the unrealistic assumptions of constant and linear growth, respectively, we utilize a general strictly concave resource growth function, which captures in a better way the essential features of renewable resources.

We demonstrate that the nature of steady state equilibrium depends on the value of the intertemporal elasticity of substitution in consumption. In particular, if the size of the intertemporal elasticity of substitution is at least half, but different from one, then stationary equilibria are saddle points. The equilibrium is stable under the logarithmic utility function when intertemporal elasticity of substitution is equal to one. For smaller values of the intertemporal elasticity of substitution we use a parametric example to demonstrate the existence of a subcritical flip bifurcation for the case of an inefficient equilibrium. This means that a repelling two-cycle emerges on the side of flip bifurcation, where the steady state is stable. Thus we obtain cycles and indeterminacy from a model with standard well-behaved utility function and constant returns to scale production function in the absence of externalities or imperfect competition.

We proceed as follows. The elements of a conventional overlapping generations economy augmented by dynamics and growth of a renewable resource is presented, and the equilibrium conditions of the economy characterized in section 2. Conditions for a unique steady state and its efficiency are described in section 3. In section 4 we study dynamic equilibria of a planar system consisting of harvesting and stock of a renewable resource, and end up with a characterization when all the stationary equilibria are saddle points. Section 5 analyzes the case of the logarithmic utility function when the intertemporal elasticity of substitution is unity. Since saddle point equilibria may not exist if the intertemporal elasticity of substitution in consumption is low enough, section 6 studies what happens in this case. A subcritical flip bifurcation is shown to occur under certain parametric constellations when the steady state displays dynamic inefficiency. Finally, section 7 summarizes our findings.

2. The Model and the Equilibrium Conditions

We consider an overlapping generations economy where agents live for two periods. There is no population growth. Agents maximize the following intertemporally additive lifetime utility function

(1) $$V = u(c_1^t) + \boldsymbol{b} u(c_2^t),$$

where c_i^t denotes the period i (=1,2) consumption of consumer-worker born at time t and $\boldsymbol{b} = (1+\boldsymbol{d})^{-1}$ with $\boldsymbol{d}$ being the rate of time preference. We assume that $u' > 0$, $u'' < 0$ and the Inada conditions, i.e. $\lim_{c \to \infty} u'(c) = 0$ and $\lim_{c \to 0} u'(c) = \infty$. The young are endowed with one unit of labor, which they supply inelastically to firms in consumption goods sector. The labor earns a competitive wage. The representative consumer-worker uses the wage to buy consumption good and to save. He can save in the financial asset or buy the available stock of the renewable resource.

The firms in the consumption good sector have a constant returns to scale technology, $F(H_t, L_t)$, to transform the harvested resource (H_t) and labor (L_t) into output. This technology can be expressed in factor intensive form to give $F(H_t, L_t)/L_t = f(h_t)$, where h_t (= H_t / L_t) is the per capita level of the harvest. The per capita production function has the standard properties: $f' > 0$ and $f'' < 0$. Furthermore, we assume $\lim_{h \to 0} f'(h_t) = \infty$ and $\lim_{h \to \infty} f'(h_t) = 0$.

The renewable resource in our model has two roles. It is both a store of value and an input in the production of consumption good. The market for the resource operates in the following manner. At the beginning of the period the old agents own the stock, and also receive that period's growth of the stock. They sell the stock (growth included) to the firms, which then decide how much of that resource to harvest and use as an input in the production of the consumption good. The firm will sell the remaining stock of the resource to the young at the end of the period. Alternatively we could think of the old deciding how much to harvest of the resource and how much to sell to the young.

The growth of the resource (the growth function) is $g(x_t)$, where x_t denotes the beginning of period t stock of the resource. $g(x_t)$ is assumed to be a strictly concave function, i.e. $g'' < 0$. Besides owning the stock the current old generation (generation t-1 in period t) will also get its growth, i.e. the stock they have available for trading is $x_t + g(x_t)$. Furthermore, we assume that there are two values $x = 0$ and $x = \tilde{x}$ for which $g(0) = g(\tilde{x}) = 0$. Consequently, there is a unique value $\hat{x}$ at which $g'(\hat{x}) = 0$. Hence, $\hat{x}$ denotes the level of stock where the growth is maximized, providing the maximum sustained yield (MSY). $\tilde{x}$ is the level at which the stock is so large that growth is zero. It is the maximal stock that the natural environment can sustain. For instance a quadratic growth function ($g(x) = ax - (1/2)bx^2$) reflecting logistic growth for renewable resources fulfills these assumptions.

The transition equation for the resource is

$$(2) \qquad x_{t+1} = x_t - h_t + g(x_t),$$

where h_t denotes that part of the resource stock which has been harvested for use as an input in production. The initial stock and its growth, $g(x_t)$, can be conserved for the next period's stock or used for this period's harvest.

In addition to trading in the resource markets, the young can also participate in the financial markets by borrowing or lending, the amount of which is denoted by s_t. The periodic budget constraints are thus

$$(3) \qquad c_1^t + p_t x_{t+1} + s_t = w_t$$

(4) $$c_2^t = p_{t+1}[x_{t+1} + g(x_{t+1})] + R_{t+1}s_t$$

where p_t is the price of the resource stock in terms of period t's consumption, w_t is the wage rate, and $R_{t+1} = 1 + r_{t+1}$ is the interest factor. The young generation buys an amount x_{t+1} of the resource stock from the representative firm. The firm has harvested an amount h_t of the stock, and x_{t+1} has been left to grow. According to (4) the old generation consumes their savings including the interest, and the income they get from selling the resource next period to the firm, $p_{t+1}[x_{t+1} + g(x_{t+1})]$.

The periodic budget constraints (3) and (4) imply the lifetime budget constraint

(5) $$c_1^t + \frac{c_2^t}{R_{t+1}} = w_t + \frac{p_{t+1}[x_{t+1} + g(x_{t+1})] - R_{t+1}p_t x_{t+1}}{R_{t+1}}$$

Maximizing (1) subject to (5) and to the appropriate nonnegativity constraints (which we do not have to worry about because of our assumptions on the utility, production and growth functions) leads to the following first-order conditions for s_t and x_{t+1}

(6) $$u'(c_1^t) = R_{t+1}\boldsymbol{b}u'(c_2^t)$$

(7) $$p_t u'(c_1^t) = p_{t+1}[1 + g'(x_{t+1})]\boldsymbol{b}u'(c_2^t).$$

These conditions have straightforward interpretations. (6) is the Euler equation which says that the marginal rate of substitution between today's and tomorrow's consumption should be equal to the interest factor. According to (7) the marginal rate of substitution between consumptions in two periods should be equal to the resource price adjusted growth factor. (6) and (7) together imply the arbitrage condition for two assets

(8) $$R_{t+1} = [1 + g'(x_{t+1})]\frac{p_{t+1}}{p_t},$$

according to which the interest factor is equal to the resource price adjusted growth factor. Using (8) we can rewrite the lifetime budget constraint as

$$(9) \qquad c_1^t + \frac{c_2^t}{R_{t+1}} = w_t + \frac{p_{t+1}\left[g(x_{t+1}) - g'(x_{t+1})x_{t+1}\right]}{R_{t+1}}.$$

The term in the square brackets is positive, since the growth function is strictly concave.

After presenting the elements of the model, we turn next to characterize the equilibria and dynamics of the model. The competitive equilibrium is defined as follows.

Definition. A sequence of a price system and a feasible allocation,

$\left\{p_t, R_t, w_t, c_1^t, c_2^{t-1}, h_t, x_t\right\}_{t=1}^{\infty}$ is a competitive equilibrium, if

(i) given the price system consumers maximize subject to their budget constraints and

(ii) markets clear for all t = 1,2,...,T,...

Market clearing conditions are

$$(10a) \quad c_1^t + c_2^{t-1} = f(h_t)$$

$$(10b) \quad x_{t+1} + h_t = x_t + g(x_t)$$

$$(10c) \quad s_t = 0$$

$$(10d) \quad f'(h_t) = p_t$$

$$(10e) \quad f(h_t) - h_t f'(h_t) = w_t$$

(10a) is the resource constraint for all t, and (10b) is the transition equation for the renewable resource stock. The fact that there is only one type of a consumer per generation and no government debt forces the asset market clearing condition to be such that saving $s_t = 0$ for all t. Equations (10d) and (10e) in turn are the first-order conditions for profit maximization, and determine the evolution of prices, p_t and w_t.

Market clearing condition (10b) and the first-order condition (7) for the resource stock and harvesting imply the following planar system that describes the dynamics of the model.

(11) $$x_{t+1} = x_t - h_t + g(x_t)$$

(12) $$f'(h_t)u'\left[f(h_t) - f'(h_t)h_t - f'(h_t)x_{t+1}\right] =$$
$$b\, f'(h_{t+1})u'\left[f'(h_{t+1})(x_{t+1} + g(x_{t+1}))\right]\ \left[1 + g'(x_{t+1})\right]$$

We have used the periodic budget constraints (3) and (4), and the equilibrium conditions (10d) and (10e), to arrive at equation (12). Equations (11) and (12) are the main objects of our study.[5] Before analyzing the qualitative properties of this system we characterize the stationary equilibrium.

3. Stationary Equilibria and Efficiency

In the steady states ($\Delta h_t = 0$ and $\Delta x_t = 0$) the following equations hold

(13) $$h = g(x)$$

(14) $$u'\left[f(h) - f'(h)h - f'(h)x\right] = b\, u'\left[f'(h)(x + g(x))\right]\ \left[1 + g'(x)\right].$$

Given the properties of the growth function, the curve defined by (13) is not monotonic. Totally differentiating (14) we get

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[5] Instead of using Euler equation (12) we could have taken another route for the dynamic analysis by concentrating on the evolution of savings defined as $q(w_t, R_{t+1}, p_{t+1}, p_t) \equiv w_t - c_1^t(w_t, R_{t+1}, p_{t+1}, p_t)$. It is straightforward to show that $\partial q / \partial R_{t+1} < 0$, when the intertemporal elasticity of substitution is less than unity. See discussion below on the crucial importance of this elasticity in our analysis.

(15) $$\frac{dh}{dx} = \frac{b\,u'(c_2)g''+b\,u''(c_2)f'(1+g')^2 + u''(c_2)f''}{-u''(c_1)f''(x+h) - b\,f''(x+h)} > 0 .$$

This means that the stationary Euler equation is an increasing curve in the hx-space. Next we show that the curve defined by (14) goes through the origin in the hx-space.

Lemma. The point $\{h = 0, x = 0\}$ fulfills equation (14).

Proof. Suppose the Euler equation does not go through the origin. Since the curve is upward sloping, there are two possibilities for the limiting behavior. First, if we let $x \to 0$, then h must go towards some positive number. Secondly, if we let $h \to 0$, then x must approach some positive number. In the first case the right-hand side of (14) approaches infinity (if $g'(x)$ approaches infinity when x approaches zero, this effect will reinforce the argument), because $\lim_{c\to 0} u'(c) = \infty$, but the left-hand side approaches some finite number. Thus equation (14) cannot hold. In the second case when $h \to 0$ the right-hand side approaches zero, since $\lim_{c\to \infty} u'(c) = 0$, but the argument for the left-hand side (the first period consumption) approaches a negative number, which is not a feasible solution to the consumer's optimization problem. **Q.E.D**.

It is quite straightforward to see that the steady state in our model is not necessarily unique. When the growth rate is $g'(x) > 0$, the upward sloping Euler equation can cross the growth curve in many points. For steady state to be unique, it is necessary that the Euler equation cuts the growth curve from below. If it cuts the growth curve from above, there are more than one equilibrium. For growth rate $g'(x) \leq 0$ the stationary equilibrium is necessarily unique because of decreasing resource growth curve. In the subsequent analysis we will concentrate on the nontrivial unique steady state.[6]

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[6] It can also be the case that the only point where the curves cross is the origin, especially, since we have not imposed Inada conditions on the growth function.

We will describe the loci $\Delta x_t = 0$ and $\Delta h_t = 0$ in the hx-space. The slope of the locus, $h_t = g(x_t)$, evaluated at the steady state is

$$(16) \qquad \left.\frac{dh_t}{dx_t}\right|_{\Delta x_t = 0} = g'(x).$$

The slope of the locus (derived in Appendix 1) determined by equation (12), and evaluated at the steady state is

(17)

$$\left.\frac{dh_t}{dx_t}\right|_{\Delta h_t = 0} = \frac{u''(c_1)f'(1+g') + \boldsymbol{b}u'(c_2)g''(1+g') + \boldsymbol{b}u''(c_2)f'(1+g')^3}{u''(c_1)f' - u''(c_1)f''(x+g) + \boldsymbol{b}u'(c_2)g'' - \boldsymbol{b}u''(c_2)(1+g')\left[f''(x+g) - f'(1+g')\right]}$$

The slope in (16) can be positive, zero or negative. The slope in (17) is always positive given our assumptions on the utility function and the fact that $1+g'$ needs to be always positive, because in the stationary equilibrium $1+g'$ equals the interest factor (c.f. arbitrage equation (8)).

The fact that we concentrate on the unique steady state means that the following holds in the stationary equilibrium

$$(18) \qquad \left.\frac{dh_t}{dx_t}\right|_{\Delta h_t = 0} > \left.\frac{dh_t}{dx_t}\right|_{\Delta x_t = 0}.$$

This means that Euler equation cuts the growth curve from below, see Figures 1 and 2 below.

To summarize, we have argued that a unique stationary equilibrium exists, when the growth rate, $g'(x)$, is nonpositive. In the case of positive growth rate a necessary condition for the steady state to unique is that the Euler equation cuts the growth curve from below. There are multiple equilibria if $g'(x)$ is positive and Euler equation cuts the resource growth curve from above.

Are the stationary equilibria efficient? It is a well-known fact that the competitive equilibria in overlapping generations models can be inefficient. Keeping in mind that $g'(x)$ is the rate of interest in the steady state and the population growth rate is zero in our model, we conclude that all those steady states for which $g'(x) \geq 0$ are efficient. This is the case where the real interest rate exceeds population growth rate.

Steady states in which $g'(x) < 0$ are inefficient, since consumption could be increased for every generation by harvesting some of the resource stock during any period. This case corresponds to the situation where the real interest rate is less than the population growth rate. This overaccumulation is inefficient.[7]

4. Dynamical Equilibria: Saddles

To study the qualitative properties of our model we start by considering paths for which $x_{t+1} \geq x_t$ and $h_{t+1} \geq h_t$. It follows from (11)

$$(19) \qquad x_{t+1} \geq x_t \Leftrightarrow x_t - h_t + g(x_t) \geq x_t \Leftrightarrow g(x_t) \geq h_t .$$

This means that x is increasing below the growth curve, and it is decreasing above the curve.

Considering paths for which $h_{t+1} \geq h_t$, requires more work. In Appendix 1 (equation A.3) we derive the following expression (evaluated at the steady state) for the derivative of the right-hand side of equation (12) above with respect to h_{t+1} (denoted also by A)

$$(20) \qquad \frac{\partial RHS}{\partial h_{t+1}} = (1+g')\, b f'' u' \left(1 - \frac{1}{r(c_2)}\right) \equiv A ,$$

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[7] Efficiency outside steady states is a more involved problem. One can study the efficiency of nonstationary paths by modifying the criterion developed by Cass (1972) to the needs of the model at hand.

where $\rho(c)$ $(= -[u'(c)/cu''(c)])$ is the reciprocal of the elasticity of the marginal utility of consumption. This quantity is also known as the intertemporal elasticity of substitution, and it depends inversely on the curvature of the periodic utility function. We can see that given the values of x_t and h_t, the right-hand side of equation (12) is an increasing (decreasing) function of h_{t+1}, if ρ is less (greater) than unity.[8]

If $\rho > 1$ we get from (12)

$$
\begin{aligned}
(21) \qquad & h_{t+1} \geq h_t \Leftrightarrow f'(h_t)u'[f(h_t) - f'(h_t)h_t - f'(h_t)x_{t+1}] \leq \\
& \beta f'(h_t)u'[f'(h_t)(x_{t+1} + g(x_{t+1}))]\ [1 + g'(x_{t+1})]
\end{aligned}
$$

Equation (21) is equivalent to the following statement

$$
(22) \qquad \frac{u'[f(h_t) - f'(h_t)h_t - f'(h_t)x_{t+1}]}{\beta u'[f'(h_t)(x_{t+1} + g(x_{t+1}))]\ [1 + g'(x_{t+1})]} \leq 1
$$

If $\rho < 1$, the inequalities in (21) and (22) are reversed. All this means that the motion of h on both sides of the curve, where $h_{t+1} = h_t$, depends on the value of intertemporal elasticity of substitution. This fact points out to the possibility that dynamics of the system can drastically change when ρ passes through unity. When $\rho = 1$, the preferences are logarithmic. We will return to this case later on in section 5.

The crucial role of ρ is illustrated in Figures 1 and 2. In Figure 1, where the intertemporal elasticity of substitution is greater than one, the arrows indicate a possibility of saddle point equilibrium.[9] In this section we give a formal proof for this intuition. In Figure 2, where the intertemporal elasticity of substitution is less one, the arrows describing the motion

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[8] When the utility function belongs to the class of constant relative risk aversion (CRRA) functions, the inverse of the relative risk aversion measure is the intertemporal elasticity of substitution. See e.g Deaton (1991) for a further discussion.

[9] The direction of h on both sides of the hh curve in diagrams 1 and 2 can be obtained as follows. Consider equation (21) as an equality. Differentiate both sides with respect to h keeping x fixed. E.g . in the case of $\rho > 1$, the left-hand-side decreases and the right-hand side increases, which means that above the curve, h is increasing and below it is decreasing (c.f. equation (21) again). Analogously, it can be shown that the direction of the arrows is reversed when ρ <1.

of harvesting are reversed. This suggests a possibility for the stable equilbrium. One should notice, however, that orbits in discrete dynamical systems are sequences of points in the relevant state spaces. This qualitative information drawn from discrete phase diagrams is quite tentative and must be confirmed analytically, which we will do in detail in the next section.

In order to study formally the stability properties of dynamical equilibrium, we first rewrite equation (11) as follows

$$x_{t+1} = x_t - h_t + g(x_t) \equiv G(x_t, h_t) \tag{23}$$

Substituting the RHS of (11) for x_{t+1} in (12) gives an implicit equation for h_{t+1},

$$h_{t+1} = F(x_t, h_t) \tag{24}$$

The planar system describing the dynamics of the renewable resource stock and harvesting consists now of equations (23) and (24). The Jacobian matrix of the partial derivatives of the system (11)-(12) can be written as

$$J = \begin{bmatrix} G_x & G_h \\ F_x & F_h \end{bmatrix} = \begin{bmatrix} 1+g' & -1 \\ \frac{C}{A} & \frac{B}{A} \end{bmatrix}, \tag{25}$$

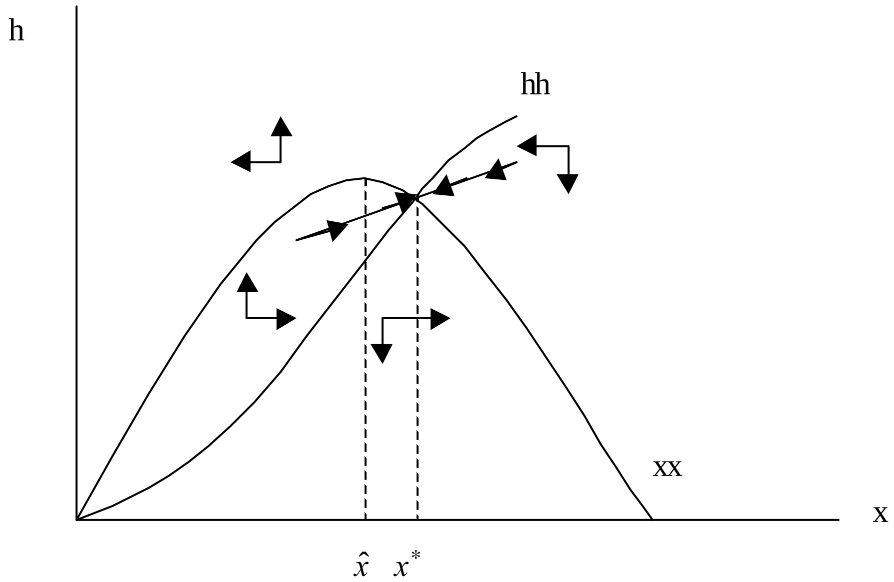

Figure 1. Elasticity of intertemporal substitution greater than one

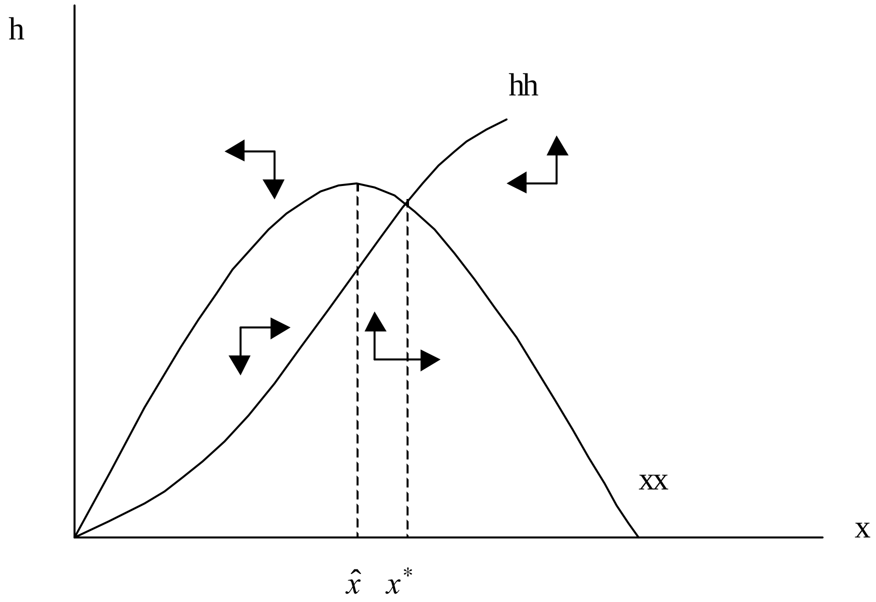

Figure 2. Elasticity of intertemporal substitution less than one

where A has been derived above in equation (20) and B and C are the partial derivatives of equation (12) with respect to h_t and x_t respectively, and have been derived in Appendix 2. By defining $\hat{r} = \frac{r}{r-1}$ the two ratios in the Jacobian matrix can then be expressed as

$$(26) \qquad \frac{C}{A} = \left\{ -\frac{f'^2 u''(c_1)}{\mathbf{b} f''u'(c_2)} - \frac{f'^2 u''(c_2)(1+g')^2}{f''u'(c_2)} - \frac{f'g''}{f''} \right\} \hat{r}$$

$$(27) \qquad \frac{B}{A} = \left\{ 1 - \frac{f'u''(c_1)(x+h)}{u'(c_1)} + \frac{f'^2 u''(c_1)}{f''u'(c_1)} + \frac{f'^2 (1+g')u''(c_2)}{f''u'(c_2)} + \frac{f'g''}{f''(1+g')} \right\} \hat{r} \,,$$

where we can see the importance of the magnitude of the intertemporal elasticity of substitution for the stability analysis. These elements of the Jacobian change signs whenever r passes through unity, since the bracketed term in C/A is negative and in B/A is positive.

The trace and determinant of the characteristic polynomial of our system can be calculated as

$$(28) \qquad D = (1+g')\hat{r} \left\{ 1 - \frac{f'u''(c_1)(x+h)}{u'(c_1)} \right\}$$

(29)

$$T = (1+g') + \left\{ 1 - \frac{f'u''(c_1)(x+h)}{u'(c_1)} + \frac{f'^2 u''(c_1)}{f''u'(c_1)} + \frac{f'^2 (1+g')u''(c_2)}{f''u'(c_2)} + \frac{f'g''}{f''(1+g')} \right\} \hat{r} \,.$$

Armed with these calculations (see Appendix 2 for details) we get the following Proposition

Proposition 1. If the intertemporal elasticity of substitution is at least one half, and differs from unity, all the stationary equilibria are saddle points.

Proof. See Appendix 3.

According to Proposition 1, stationary equilibria are saddle points for a wide range of the values for the intertemporal elasticity of substitution. Empirical evidence on the size of this elasticity does not, however, necessarily coincide with these parameter values, but often points out to lower values.[10] It is therefore of interest to study also the characteristics of equilibria of the special case when $r = 1$ and when $r < 1/2$. These equilibria are studied in the next two sections.

5. Dynamical Equilibria under the Logarithmic Utility Function: Stability

Next we consider the case, where the intertemporal elasticity of substitution is unity, i.e. the periodic utility function is logarithmic, $u(c) = \ln c$. In this case (12) can be written as

$$(30) \qquad \frac{f'(h_t)}{f(h_t) - h_t f'(h_t) - f'(h_t)x_{t+1}} = \frac{b\left[1 + g'(x_{t+1})\right]}{x_{t+1} + g(x_{t+1})}.$$

Using (11) in (30) gives a relation between h_t and x_t, defined as $h_t = P(x_t)$. Hence h_{t+1} disappears from the Euler equation (12) so that our planar system (11)-(12) is reduced to a first-order nonlinear difference equation for x

$$(31) \qquad x_{t+1} = x_t - P(x_t) + g(x_t).$$

Once the evolution of x is determined, the behavior of h can be obtained from (12) so that the system has become recursive. What are the dynamic properties of this system?

The slope of the first-order nonlinear difference equation (31) is

$$(32) \qquad \frac{dx_{t+1}}{dx_t} = 1 - P'(x_t) + g'(x_t).$$

---

[10] See the discussion e.g. in Deaton (1991, pp. 63-75).

In order to develop the expression for $P'(x_t)$ we take into account (11) and rewrite (30) as

$$
\begin{aligned}
(33)\qquad & f'(h)\left[x-h+g(x)+g(x-h+g(x))\right]= \\
& \mathbf{b}\left[1+g'(x-h+g(x)\right]\left[f(h)-f'(h)x-f'(h)g'(x)\right].
\end{aligned}
$$

We totally differentiate (33) with respect to h and x, and define $\bar{x}=x-h+g(x)$ to get

(34)

$$\left\{f''\left[\bar{x}+g(\bar{x})\right]-f'(1+g'(\bar{x}))+\mathbf{b}\,g''(\bar{x})\left[f-f'(x+g(x))\right]-\mathbf{b}\,(1+g'(\bar{x}))\left[f'-f''(x+g(x))\right]\right\}dh$$

$$=(1+g'(x))\left\{\mathbf{b}\,g''(\bar{x})\left[f-f'(x+g(x))\right]-\mathbf{b}\,(1+g'(\bar{x}))f'-f'(1+g'(x))\right\}dx$$

Denoting the term in braces on the left-hand side by Y and on the right-hand side by Z we get

$$(35)\qquad \frac{dh}{dx}=P'(x)=\frac{\left[1+g'(x)\right]\left\{Z\right\}}{\left\{Y\right\}}>0,$$

since both Y and Z are negative. Next we evaluate (35) in the steady state, where $x=\bar{x}$. Note that now $Y=Z+f''(x+g)\left[1+\mathbf{b}\,(1+g')\right]$, which means that $0<P'(x)<1+g'$, because $\left|\frac{Z}{Y}\right|<1$. From here it follows that $1-P'(x)+g'(x)>0$.

To prove the stability of the steady state we need to have $1+g'(x)-P'(x)<1 \quad \Leftrightarrow P'(x)>g'(x)$. This condition holds for all inefficient equilibria (where $g'(x)\leq 0$), which are thus stable. What about the stability of efficient equilibria (where $g'(x)>0$)? If the stationary equilibrium is unique, then the upward sloping Euler equation cuts the resource growth curve from below so that the inequality (18) holds. This is equivalent to the stability condition $P'(x)>g'(x)$. Hence we can summarize our findings in

Proposition 2. Under the logarithmic utility function, the planar system *reduces* to a nonlinear first-order difference equation for the natural resource stock. If the stationary equilibrium is unique, it is stable regardless of whether the equilibrium is efficient or not.

As Proposition 2 reveals, the logarithmic utility function is very special. When the intertemporal elasticity of substitution becomes unity, the planar system turns to a first-order nonlinear difference equation for the resource stock, saddle point equilibria vanish and stable equilibria emerge. Next we turn to examine the case, where $r < 1/2$.

6. Dynamical Equilibria: Indeterminacy and Flip Bifurcations

In the above discussion we found that when $r > 1$, the determinant (D) and the trace (T) of the system are positive, and furthermore that D-T+1 < 0. Stationary equilibria are thus saddles. These equilibria are in area X in Figure 3 in which we have reproduced the familiar graphical description of dynamical equilibria in a planar system (see e.g. Azariadis 1993). Stable equilibria lie in area B, and the other saddle point equilibria are in area A. Thus complex roots are not possible in our model, which in turn means that we cannot get Hopf bifurcations.

When $r < 1$, the determinant of the system becomes negative, and D-T+1 positive. This means that the saddle-node bifurcations (they require among other things that D-T+1 = 0) are not possible. We already proved that stationary equilibria are saddles for $1 > r \geq 1/2$. Since D+T+1 cannot be unambiguously signed for $r < 1/2$, it is possible to have flip bifurcations in our model (see areas A and B in Figure 3).

In the following we assume $r < 1/2$ (i.e. $\hat{r} < 0$ and $|\hat{r}| < 1$). Inspecting the general case above seems to point out to the fact that it is possible to get stable equilibria and flip bifurcations. Since $\hat{r} < 0$, we consider the case where D < 0. We have also established in the proof of Proposition 1 that, when $r \geq 1/2$ (and $r \neq 1$) D-T+1 > 0. To get stability, we need to have D+T+1 > 0 as well. Because we have rigorously shown the existence of saddles when D < 0, we can also show the existence of flip bifurcations, if we can show the stability of equilibria.

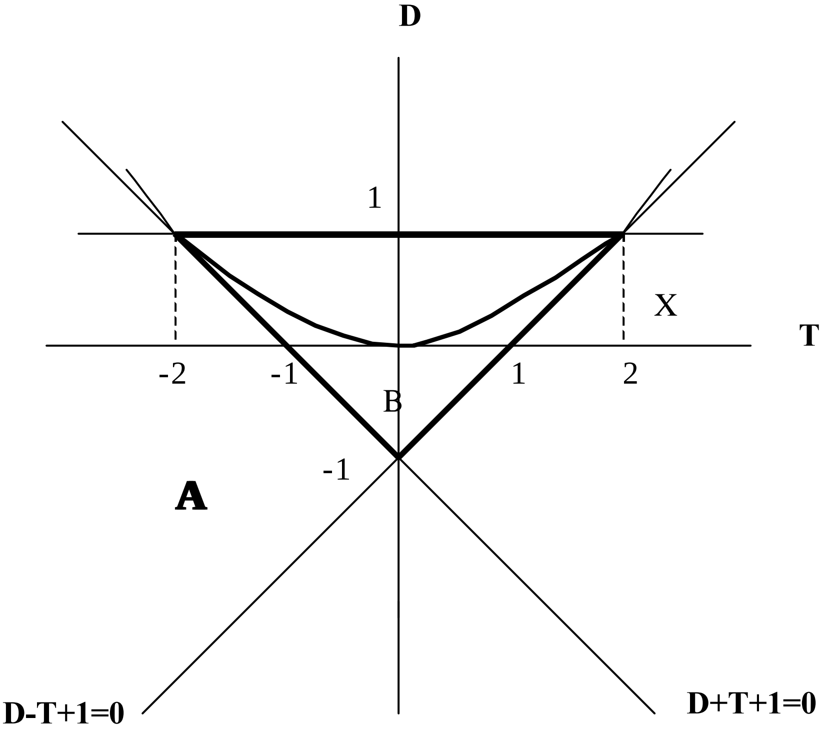

Figure 3. Characteristics of stability in a planar system

To proceed we rewrite D+T+1 as follows

(36) $$D = (1+g')\,\hat{r}\{M+1\}$$

(37) $$T = (1+g') + \hat{r}\{M+N+1\},$$

where

$$M = -\frac{f'u''(c_1)(x+h)}{u'(c_1)} > 0$$

$$N = \left\{\frac{f'^2\,u''(c_1)}{f''u'(c_1)} + \frac{f'^2\,(1+g')u''(c_2)}{f''u'(c_2)} + \frac{f'\,g''}{f''(1+g')}\right\} > 0\,.$$

Using this shorthand notation we can express D+T+1 after some manipulation

(38) $$\text{D+T+1} = (2+g')\,\hat{r}M + \hat{r}N + (2+g')(1+\hat{r})\,.$$

This shows that at least in principle D+T+1 can be zero or positive, if the last term, the only positive term in the expression, dominates. Note that when D <0, D-T+1 > 0 and D+T+1 = 0 we have a flip bifurcation (see the line between areas A and B).

Since the existence of stable equilibria (indeterminacy) and a flip bifurcation cannot be proved analytically in our model we consider a parametric example. We use the following standard explicit functional forms:

$$\begin{cases} u(c) = \dfrac{c^{1-\frac{1}{r}}}{1-\dfrac{1}{r}} \Rightarrow u'(c) = c^{-\frac{1}{r}}, u''(c) = -\dfrac{1}{r}c^{-\frac{1}{r}-1} \\ f(h) = h^{a} \Rightarrow f' = ah^{a-1}, \quad f'' = a(a-1)h^{a-2} \\ g(x) = ax - \dfrac{1}{2}bx^2 \Rightarrow g' = a - bx, \quad g'' = -b, \; 1+g' = 1+a-bx \end{cases}$$

Note that r in the utility function is exactly the intertemporal elasticity of substitution. In the stationary equilibrium $h = ax-(1/2)bx^2$. Using this expression for h, the Euler equation and budget constraints, we end up with the following expression (see Appendix 4) for the stock of the renewable resource in a stationary equilibrium

(39) $$\frac{1}{1+(1+a-bx)^r b^r} + \frac{a}{a-\dfrac{1}{2}bx} = 1-a\,.$$

A straightforward but tedious calculation yields the expression for D+T+1

(40) $$D+T+1=\left(\frac{1}{\rho-1}\right)\frac{(2+a-bx)\alpha(1+a-\frac{1}{2}bx)}{\left[(1-\alpha)(a-\frac{1}{2}bx)-\alpha\right]}+(2+a-bx)\left(\frac{1-2\rho}{1-\rho}\right)$$

$$+\left(\frac{1}{1-\alpha}\right)\left(\frac{1}{\rho-1}\right)\left[\alpha\left[1+(1+a-bx)^{r}\beta^{\rho}\right]+\frac{\alpha(1+a-bx)^{1-r}\left[1+(1+a-bx)^{r}\beta^{\rho}\right]}{\beta^{\rho}}+\frac{\rho b(ax-\frac{1}{2}bx^{2})}{1+a-bx}\right]$$

In the sequel we undertake a numerical analysis for a calibrated version of the parametric example of our model. We assume the following values for parameters of the growth function and the discount factor: $a=b=1$ and $\beta=1/2$.[11] The values for growth parameters mean that $\hat{x}=1$ and $\tilde{x}=2$, and furthermore that the condition $1+g'(x)\geq 0$ holds for all $0\leq x\leq 2$. Economically more interesting parameters are the output elasticity of resource (α), which determines the price elasticity of resource demand, and the intertemporal elasticity of substitution (ρ). For this reason our focus will be to find out for what values of these parameters we will get stability and flip bifurcations.

Solving α from equation (39) and plugging that value into (40) we find out for what combinations of x and ρ D+T+1 is greater or less than zero or exactly zero. Solving α from (39) we get

(41) $$\alpha=\frac{2a-bx}{2+2a-bx}-\frac{2a-bx}{2+2a-bx}\left(\frac{1}{1+(1+a-bx)^{r}\beta^{\rho}}\right).$$

Plugging this relationship (41) into (40) gives the following relatively complicated expression

(42) $$D+T+1=\left(\frac{1}{\rho-1}\right)(2+a-bx)(1+a-bx)^{r}\beta^{\rho}+(2+a-bx)\left(\frac{1-2\rho}{1-\rho}\right)$$

---

[11] If we want to interpret literally the length of the period in our overlapping generations economy to be around 25 years, then the annual discount factor 0.975 (or the rate of time preference about 2.6 percent) means that the discount factor for 25 years should be around ½.

$$+\left(\frac{1}{r-1}\right)\left[\frac{r bx(2a-bx)(2+2a-bx)\left[1+(1+a-bx)^{r} b^{r}\right]}{2(1+a-bx)\left[2a-bx+2\left[1+(1+a-bx)^{r} b^{r}\right]\right]}\right]$$

$$+\left(\frac{1}{r-1}\right)\left[\frac{(2a-bx)(1+a-bx)^{r} b^{r}(1+(1+a-bx)^{r} b^{r})}{2a-bx+2\left[1+(1+a-bx)^{r} b^{r}\right]}\right]+$$

$$+\left(\frac{1}{r-1}\right)\left[\frac{(2a-bx)(1+a-bx)(1+(1+a-bx)^{r} b^{r})}{2a-bx+2\left[1+(1+a-bx)^{r} b^{r}\right]}\right].$$

To get a more precise idea where to look for stable equilibria, note that the only positive term in this expression is the second term. Combining this term and the first term we get after rearranging

$$(43)\qquad \left(\frac{2+a-bx}{1-r}\right)\left[(1-2r)-(1+a-bx)^{r} b^{r}\right].$$

As we have already mentioned, we assume that $b = 1/2$ and $0 < r < 1/2$. Consider first the efficient allocations, which lie on the left-hand side of the maximum sustained yield, i.e. $0 \le x \le a/b$. It is quite straightforward to see that the term in the brackets of (43) is negative. This means that all the stationary equilibria are saddles. Therefore, we should look for possible stable equilibria from the right-hand side of the MSY, where equilibrium is inefficient.

The stationary equilibrium condition (39) indicates that there is an inverse relationship between a and x. Because we will now concentrate on such allocations for which $x > a/b$, the value of a must be relatively small for equation (39) to hold.

Our approach will be the following. We will first graph the plane defined by equation (42) in the (D+T+1)xr - space. Then we set D+T+1 = 0, and graph those values of x and r for which D+T+1 = 0 holds.

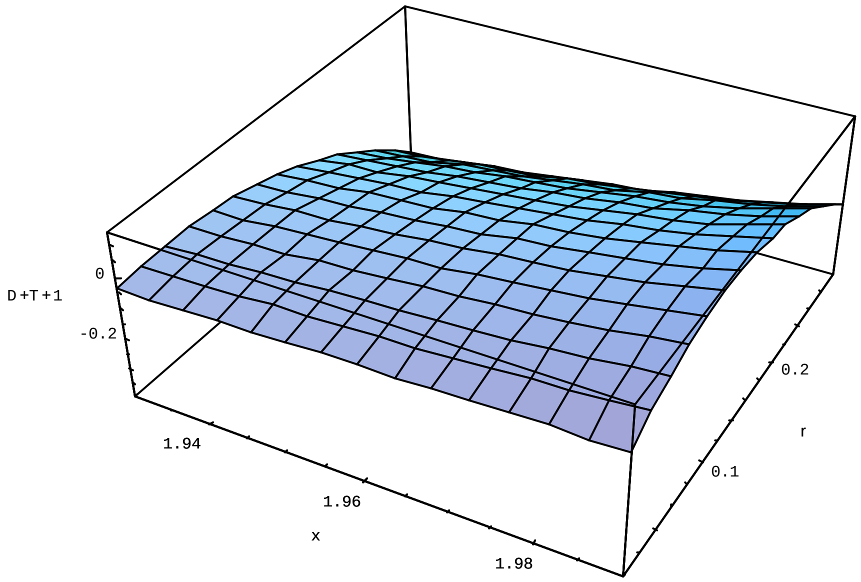

Figure 4. D+T+1.

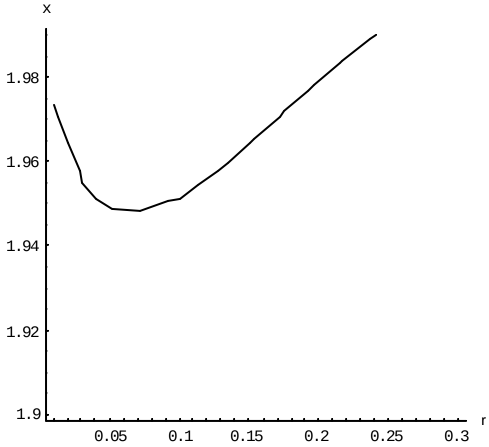

Figure 5. A characterization of indeterminacy and flip bifurcations

Figure 4 is the three dimensional graph of equation (42) (when a has been substituted in for the expression of D+T+1). It points out to the fact that D+T+1 will be positive only for extremely high (i.e. values which are close to $\tilde{x}$ (= 2) levels of the renewable resource stock.

In Figure 5 we have projected those values of the resource stock x and the elasticity of intertemporal substitution r for which D+T+1 is exactly zero, i.e., for which we have flip bifurcations. Values of x and r, which lie above the curve will yield stable equilibria, and for the values of x and r below the curve we have saddle point equilibria.

In Figure 6 we have depicted a, x and r in the same diagram, i.e. we have graphed equation (41). This figure indicates that to get stable equilibria and flip bifurcations the value of a needs to be quite small. E.g. if $a = 0.01$ and $r = 0.03$ we get the level of the stationary equilibrium stock to be 1.95664 and the level of harvesting 0.04242. We also get $D + T + 1 = 0.00119886$. And if we let $a = 0.011$, we get the equilibrium stock to be 1.95228, the level of harvesting 0.04658, and $D + T + 1 = -0.00373852$.

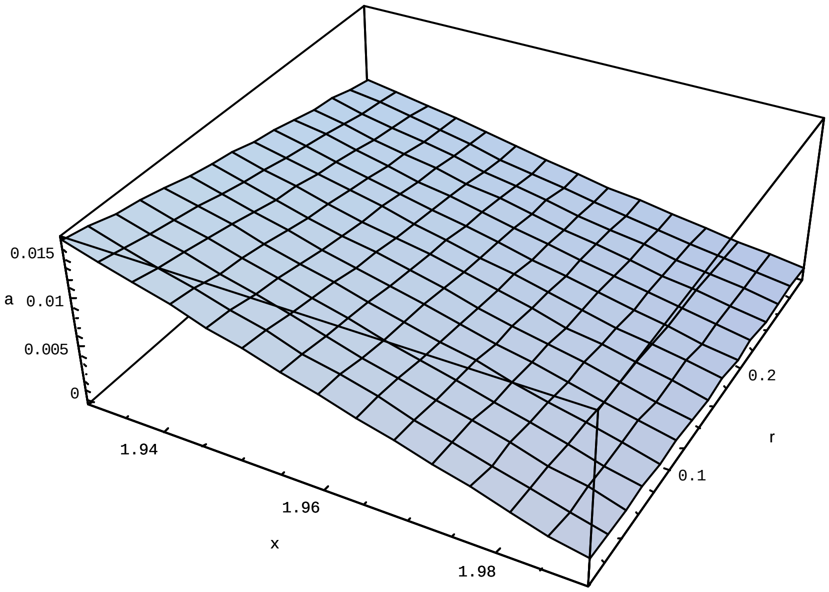

Figure 6. Equation (34).

We have shown that there is a nontrivial set of parameter values of a and r, for which our parametrized economy exhibits stable equilibria, i.e., indeterminacy, and flip bifurcations. In contrast to previous literature, indeterminacy arises here under standard assumptions on utility and production functions. This also means that there can be endogenous cycles in our model, since the characteristic roots are of different sign.[12]

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[12] Interestingly, Grandmont (1985) has shown in a different overlapping generations model with money that a succession of flip bifurcations may occur when the Arrow-Pratt relative risk version of the old agents exceeds two, which is equivalent to the condition that the intertemporal elasticity of substitution

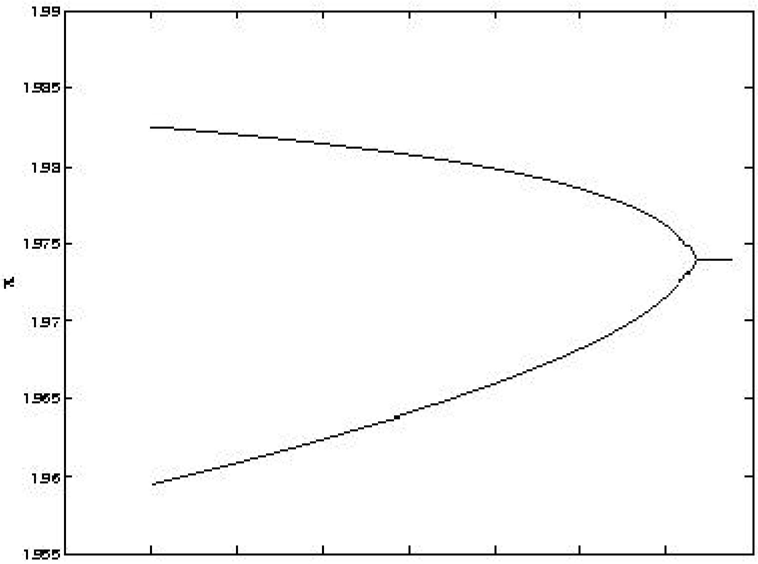
Flip bifurcations are called *period-doubling* because they give rise to stable periodic orbits whose period is twice that of the stability losing equilibrium. There are two cases to consider regarding stability. In a supercritical flip bifurcation a stable two-cycle emerges on the side of the bifurcation value where the steady state is saddle. In a subcritical flip bifurcation, an unstable two-cycle emerges on the side of the bifurcation value where the steady state is stable.[13]

To investigate the type of flip bifurcation, i.e. on which side of the flip bifurcation a two-cycle exists in our model, we resort to numerical simulations. We asked whether it is possible to find four numbers $\{x_1, h_1; x_2, h_2\}$, which solve the following transition and Euler equations.

(44) $$x_2 = x_1 - h_1 + x_1 - \frac{1}{2}x_1^2 \quad \text{and} \quad \frac{h_1^{a-1}}{\left[(1-a)h_1^{a} - a\,h_1^{a-1}x_2\right]^{\frac{1}{r}}} = \frac{b\,h_2^{a-1}(2-x_2)}{\left[a\,h_2^{a-1}(x_1+h_2)\right]^{\frac{1}{r}}}$$

(45) $$x_1 = x_2 - h_2 + x_2 - \frac{1}{2}x_2^2 \quad \text{and} \quad \frac{h_2^{a-1}}{\left[(1-a)h_2^{a} - a\,h_2^{a-1}x_1\right]^{\frac{1}{r}}} = \frac{b\,h_1^{a-1}(2-x_1)}{\left[a\,h_1^{a-1}(x_2+h_1)\right]^{\frac{1}{r}}}$$

If we find a four-tuple that fulfills the equations (44)-(45), then a two-cycle exists. We fixed $a = 0.004$, and chose the values of the intertemporal elasticity of substitution from both sides of the Flip bifurcation curve in Figure 5. In Figure 7 we have chosen to depict the emergence of the two-cycle for the resource stock x (the vertical axis). The same phenomenon happens, of course, to the level of harvesting, which we have not depicted. The flip bifurcation occurs for values of r (the horizontal axis) between 0.1825 and 0.1826. If r = 0.1826, we have a saddle, and if it is 0.1825 we have a stable equilibrium.



which means that we

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g function of the rate of

cular Guckenheimer and specific types.

Figure 7. Subcritical flip bifurcation

In overlapping generations models with one dimensional dynamics flip bifurcations have been shown to exist when the elasticity of intertemporal substitution has been less than one (see footnote 12), which also means that saving has been a decreasing function of the rate of interest (backward bending offer curves). The logistic growth function is a typical example of a simple dynamical system, which allows for complex dynamics.

Although there are no nonconvexities and market imperfections, except the typical overlapping generations structure, in our model, it seems that the case that flip bifurcations and complex dynamics emerge in our model due to the mixture of low elasticity of substitution in consumption and logistic growth.

The parameter values for the intertemporal elasticity of substitution for which we get stability and flip bifurcations are empirically quite plausible. The parameter values for the production function parameter (a), for which we obtain stability and bifurcations, are quite small. The parameter a measures the share of natural resources in total output. It varies across countries and can be relatively low.

7. Conclusions

The stability properties of an overlapping generations model with capital accumulation, like periodic equilibria and indeterminacy of equilibria, have been subject to a fairly large amount of research since the mid 1980s. These issues have not, however, been studied carefully in models with renewable resource use, like forestry or fisheries. Our purpose in this paper has been to do just that. We have examined the dynamic properties of an overlapping generations economy under the standard assumptions about the utility and production functions, but augmented with a renewable resource. In addition to a factor of production it serves as a store of value. Because a renewable resource has its own growth function and dynamics, we get a planar system consisting of harvesting and the resource stock. After having characterized the steady state equilibrium and efficiency we turned our main focus to studying the stability properties of our model.

We showed that the nature of the steady state equilibrium depends on the value of intertemporal elasticity of substitution in consumption. In particular, if the intertemporal elasticity of substitution is at least one half, but different from unity, then stationary equilibria are saddle points, but the stationary equilibrium is stable under the logarithmic utility function with the intertemporal elasticity of substitution being equal to unity. Interestingly, for smaller values of the intertemporal elasticity of substitution, which are equally plausible on the basis of empirical evidence from consumption behavior, we use a parametric example to demonstrate the existence of a subcritical flip bifurcation for the case of inefficient equilibrium. This means that a repelling two-cycle emerges on the side of flip bifurcation where the steady state is stable. Hence, an overlapping generations economy with a renewable resource may display cycles and indeterminacy under standard well-behaved utility and constant returns to scale production functions without externalities or imperfect competition as is usually required to get bifurcations and indeterminacy from stability analyses.

Appendix 1. The slope of equation [17] and the RHS of equation [12] as a function of h_{t+1}

- **The right-hand side of equation (12) as a function of h_{t+1}.**

The RHS of (12) is

A.1 $$RHS\,(h_{t+1}) = \mathbf{b}\, f'(h_{t+1})u'\left[f'(h_{t+1})(x_{t+1} + g(x_{t+1}))\right]\ \left[1 + g'(x_{t+1})\right]$$

Differentiating this with respect to h_{t+1} we get (dropping the arguments)

A.2 $$\begin{aligned} RHS'(h_{t+1}) &= (1+g')\mathbf{b}\, f''u' + (1+g')\mathbf{b}\, f'u''(c_2)\left[f''(x+g(x))\right] \\ &= (1+g')\mathbf{b}\, f''\left[u' + f'(x+g(x))u''\right] \end{aligned}$$

Keeping in mind that $c_2 = f'(x+g(x))$ we get

A.3 $$RHS'(h_{t+1}) = (1+g')\mathbf{b}\, f''u'\left(1 - \frac{1}{r(c_2)}\right)$$

where $r(c) = \dfrac{-u'(c)}{cu''(c)}$. In the case of constant Arrow-Pratt relative risk aversion utility functions $r(c)$ is exactly the elasticity of intertemporal substitution. From A.3 it is now easy to see that $RHS'(h_{t+1}) > 0 (< 0)$ when $r(c) > 1 (< 1)$.

- **The derivation of the slope of equation (17)**

We first rewrite equation (12), and take into account the fact that we consider paths, where $h_{t+1} = h_t$ for all t but x_t may vary.

A.4 $$u'\left[f(h_t) - f'(h_t)h_t - f'(h_t)x_{t+1}\right] = \mathbf{b}u'\left[f'(h_t)(x_{t+1} + g(x_{t+1}))\right](1 + g'(x_{t+1}))$$

Totally differentiating A.4 and taking into account equation (10) we get

A.5 $$\begin{aligned} &\{\ u''(c_1^t)\left[-f''(x_t + g(x_t)) + f'\right] + \mathbf{b}u'(c_2^t)g''(x_{t+1}) - \\ &\mathbf{b}u''(c_2^t)\left[f''(x_{t+1} + g(x_{t+1})) + f'(1 + g'(x_{t+1})\right](1 + g'(x_{t+1}))\ \}dh_t \\ &= \\ &\{\ u''(c_1^t)f'(1 + g'(x_t) + \mathbf{b}u'(c_2^t)g''(x_{t+1})(1 + g'(x_t)) + \\ &\mathbf{b}u''(c_2^t)\left[f'\left[1 + g'(x_t) + g'(x_{t+1})(1 + g'(x_t))\right]\right](1 + g'(x_{t+1}))\ \}dx_t\,. \end{aligned}$$

Rearranging and evaluating A.5 at the stationary point, $h_{t+1} = h_t$ and $x_{t+1} = x_t$, yields equation (17) in the text.

Appendix 2. Development of the Jacobian Matrix of the Partial Derivatives

For the purposes of stability analysis we develop the Jacobian matrix, its determinant and trace.

A.6 $\quad x_{t+1} = G(x_t, h_t)$

A.7 $\quad x_{t+1} = F(x_t, h_t)$

The stability of the steady-state depends on the eigenvalues of the Jacobian matrix of the partial derivatives

$$J = \begin{bmatrix} G_x & G_h \\ F_x & F_h \end{bmatrix}.$$

Calculating the partial derivatives of the Jacobian matrix we first obtain

$$G_x(x_t, h_t) = 1 + g'(x_t),\ G_h(x_t, h_t) = -1.$$

To get the partials of $h_{t+1} = F(x_t, h_t)$ we first do the implicit differentiation in the following manner

A.8 $\quad Adh_{t+1} = Bdh_t + Cdx_t,$

where A, B and C are appropriate partial derivatives to be presented in a moment. Calculating these we take into account the other dynamical equation of our system: $x_{t+1} = x_t - h_t + g(x_t)$. Given the definitions of A, B and C we will then have

$$F_x(x_t, h_t) = \frac{C}{A},\ F_h(x_t, h_t) = \frac{B}{A}.$$

As for A (as evaluated at the steady state) we get from A.3

A.9 $\quad A = (1 + g')b f''u'(c_2)\dfrac{r - 1}{r},$

where r has been defined in the text. For the future developments we define $\hat{r} = \dfrac{r}{r - 1}$. Clearly, $A > (<)0$, as $r < (>)1$. Totally differentiating (12) with respect to h_t (again taking into account the transition equation) we obtain

A.10 $$B = f''(h_t)u'(c_1^t) + f'(h_t)u''(c_1^t)\left[-f''(h_t)(x_t + g(x_t)) + f'(h_t)\right] + \\ b\left[f'(h_{t+1})\right]^2 u''(c_2^t)\left[1 + g'(x_{t+1})\right]^2 + b\, f'(h_{t+1})u'(c_2^t)g''(x_{t+1}) < 0,$$

and totally differentiating (12) with respect to x_t (again taking into account the transition equation) we have

A.11 $$C = -\left[f'(h_t)\right]^2 u''(c_1^t)\left[1 + g'(x_t)\right] - b\, f'(h_{t+1})u'(c_2^t)g''(x_{t+1})\left[1 + g'(x_t)\right] - \\ b\left[f'(h_{t+1})\right]^2 u''(c_2^t)\left[1 + g'(x_t)\right]\ \left[1 + g'(x_{t+1})\right]^2 > 0.$$

Next we evaluate A, B and C at the steady state. By taking into account the Euler condition at the steady state $u'(c_1) = (1 + g')\,b u'(c_2)$, we get

A.12i $$\frac{C}{A} = \left\{-\frac{f'^2\, u''(c_1)}{b f''u'(c_2)} - \frac{f'^2\, u''(c_2)(1 + g')^2}{f''u'(c_2)} - \frac{f'g''}{f''}\right\}\hat{r}$$

A.12ii $$\frac{B}{A} = \left\{1 - \frac{f'u''(c_1)(x + h)}{u'(c_1)} + \frac{f'^2\, u''(c_1)}{f''u'(c_1)} + \frac{f'^2\,(1 + g')u''(c_2)}{f''u'(c_2)} + \frac{f'g''}{f''(1 + g')}\right\}\hat{r}\ .$$

Clearly, $C/A > (<)0$ when $r < 1(> 1)$, and $B/A > (<)0$ when $r > 1(< 1)$.

We can now rewrite the Jacobian as follows

A.13 $$J = \begin{bmatrix} 1 + g' & -1 \\ \dfrac{C}{A} & \dfrac{B}{A} \end{bmatrix}.$$

The determinant (D) and the trace (T) of the Jacobian matrix, J, are D = $(1 + g')\dfrac{B}{A} + \dfrac{C}{A}$ and T = $1 + g' + \dfrac{B}{A}$ respectively. Using equations A.9, A.10 and A.11 we have the following expressions

A.14 $$D = (1 + g')\,\hat{r}\left\{1 - \frac{f'u''(c_1)(x + h)}{u'(c_1)}\right\}$$

A.15

$$T=(1+g')+\left\{1-\frac{f'u''(c_1)(x+h)}{u'(c_1)}+\frac{f'^2u''(c_1)}{f''u'(c_1)}+\frac{f'^2(1+g')u''(c_2)}{f''u'(c_2)}+\frac{f'g''}{f''(1+g')}\right\}\hat{r}\ .$$

Appendix 3. Proof of Saddle-Point Stability

We analyze the stability of system (23) and (24) on the basis of (11) and (12).

The characteristic polynomial associated with the system (23) – (24) expressed in terms of D and T is

A.16 $p(l)=l^2-Tl+D=0$

It is known from the stability theory of difference equations (see e.g. Azariadis, 1993, pp. 63-67) that for a saddle point the roots of $p(l)=0$ need to be on both sides of unity. Thus for a saddle we need that D-T+1 < 0 and D+T+1 > 0 or D-T+1 > 0 and D+T+1 < 0.

When $\hat{r}$ is positive, i.e. $r>1$, it is easy to conclude that both the determinant and the trace in A.14 and A.15, respectively, are positive, which also means that that D+T+1 > 0 holds. Making inferences about the sign of D-T+1 requires more work. A straightforward calculation yields

A.17 D-T+1=

$$g'(\hat{r}-1)+\hat{r}\left\{-\frac{f'u''(c_1)(x+h)g'}{u'(c_1)}-\frac{f'^2u''(c_1)}{f''u'(c_1)}-\frac{f'^2(1+g')u''(c_2)}{f''u'(c_2)}-\frac{f'g''}{f''(1+g')}\right\}.$$

A.17 cannot be signed yet for $\hat{r}>0$ (i.e. $r>1$). To get the sign of D-T+1 we use the assumption that our steady state is unique. This is assured by comparing slopes of the curves, where $h_{t+1}=h_t$ and $x_{t+1}=x_t$. We develop the condition

A.18 $$\left.\frac{dh_t}{dx_t}\right|_{\Delta h_t=0}>\left.\frac{dh_t}{dx_t}\right|_{\Delta x_t=0},$$

as

A.19

$$\frac{u''(c_1)f'(1+g')+b\,u'(c_2)g''(1+g')+b\,u''(c_2)f'(1+g')^3}{u''(c_1)f'-u''(c_1)f''(x+h)+b\,u'(c_2)g''-b\,u''(c_2)(1+g')\left[f''(x+h)-f'(1+g')\right]}>g'.$$

Multiplying both sides of A.19 by the denominator (negative sign) on the left-hand side we get

A.20 $$\begin{aligned}&u''(c_1)f'(1+g')+bu'(c_2)g''(1+g')+bu''(c_2)f'(1+g')^3<\\&u''(c_1)f'g'-u''(c_1)(x+h)f''g'+bu'(c_2)g''g'-bu''(c_2)(1+g')f''(x+h)g'\\&+bu''(c_2)(1+g')^2f'g'.\end{aligned}$$

and collecting terms A.20 can be re-expressed as

A.21 $$\begin{aligned}&u''(c_1)f'+bu'(c_2)g''+bu''(c_2)f'(1+g')^2+u''(c_1)(x+h)f''g'\\&+bu''(c_2)(1+g')f''(x+h)g'<0.\end{aligned}$$

Dividing by ($f''bu'(c_2)<0$), using Euler condition and the fact that $c_2=f'(x+h)$ yields

A.22

$$\frac{u''(c_1)f'(1+g')}{f''u'(c_1)}+\frac{g''}{f''}+\frac{u''(c_2)f'(1+g')^2}{f''u'(c_2)}+\frac{u''(c_1)(x+h)g'(1+g')}{u'(c_1)}-\frac{1}{r}\frac{(1+g')g'}{f'}>0$$

.

Now we multiply both sides by $f'/(1+g')$ (>0) to get

A.23 $$\frac{u''(c_1)f'^2}{f''u'(c_1)}+\frac{f'g''}{f''(1+g')}+\frac{u''(c_2)f'^2(1+g')}{f''u'(c_2)}+\frac{u''(c_1)(x+h)g'f'}{u'(c_1)}-\frac{1}{r}g'>0.$$

Rearranging and taking into account the definition of $\hat{r}$ yields

A.24

$$\left(\frac{\hat{r}-1}{\hat{r}}\right)g'+\left\{-\frac{f'u''(c_1)(x+h)g'}{u'(c_1)}-\frac{f'^2u''(c_1)}{f''u'(c_1)}-\frac{f'^2(1+g')u''(c_2)}{f''u'(c_2)}-\frac{f'g''}{f''(1+g')}\right\}<0.$$

If $\hat{r}>0$ (i.e. $r>1$) we get by multiplying with $\hat{r}$

A.25

$$g'(\hat{r}-1)+\hat{r}\left\{-\frac{f'u''(c_1)(x+h)g'}{u'(c_1)}-\frac{f'^2\,u''(c_1)}{f''u'(c_1)}-\frac{f'^2\,(1+g')u''(c_2)}{f''u'(c_2)}-\frac{f'g''}{f''(1+g')}\right\}<0.$$

Note that this is exactly D-T+1, which means that we have a saddle when $r > 1$.

If $\hat{r} < 0$ (i.e. $r < 1$) we get by multiplying with $\hat{r}$

A.26

$$g'(\hat{r}-1)+\hat{r}\left\{-\frac{f'u''(c_1)(x+h)g'}{u'(c_1)}-\frac{f'^2\,u''(c_1)}{f''u'(c_1)}-\frac{f'^2\,(1+g')u''(c_2)}{f''u'(c_2)}-\frac{f'g''}{f''(1+g')}\right\}>0$$

which means that D-T+1 is positive. To get a saddle in this case, we need to have D+T+1 to be negative. To explore this possibility we check the sign of D+T+1 when $\hat{r}<0$ (i.e. $r<1$). To make this calculation more transparent we rewrite D and T as follows

A.27i $D=(1+g')\hat{r}\{M+1\}$

A.27ii $T=(1+g')+\hat{r}\{M+N+1\}$,

where

$$M=-\frac{f'u''(c_1)(x+h)}{u'(c_1)}>0$$

$$N=\left\{\frac{f'^2\,u''(c_1)}{f''u'(c_1)}+\frac{f'^2\,(1+g')u''(c_2)}{f''u'(c_2)}+\frac{f'g''}{f''(1+g')}\right\}>0.$$

Using this shorthand notation D+T+1 can be expressed after some manipulation

A.28 D+T+1 = $(2+g')\hat{r}M+\hat{r}N+(2+g')(1+\hat{r})$.

Note that, we are now considering the case, where $\hat{r} < 0$ (i.e. $r < 1$). The first two terms in (A28) are negative. The third term is also negative when $1+\hat{r}<0$. This happens when $r>1/2$. So we have a saddle in this case, too. This completes the proof of Proposition 1. **Q.E.D.**

Appendix 4. Derivation of equation (39)

Given the assumed functional forms, the Euler equation can be written

A.29 $c_2=[(1+g')b]^r\,c_1$.

Plugging this into the equilibrium condition, $c_1 + c_2 = f(h)$ and using the budget constraint $c_2 = f'(h)(x + g(x))$ gives

$$c_1 = \frac{\left[x(a-(1/2)bx)\right]^{a}}{1+(1+a-bx)^{r}\,\boldsymbol{b}^{r}} \quad \text{and} \quad c_2 = \frac{\boldsymbol{a}\left[x(a-(1/2)bx)\right]^{a}\left[1+a-(1/2)bx\right]}{(a-(1/2)bx)}.$$

If we plug these expressions for consumption back into the equilibrium condition we get equation (32) in the text.

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