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On the Optimal Use of Correlated Information in Contractual Design Under Limited Liability

Abstract

Riordan and Sappington (JET, 1988) show that in an agency relationship in which the agent's type is correlated with a public ex post signal, the principal may attain first best (full surplus extraction and efficient output levels) if the agent is faced with a lottery such that each type is rewarded for one signal realization and punished equally for all the others. Gary-Bobo and Spiegel (RAND, 2006) show that this kind of lottery is most likely to be locally incentive compatible when the agent is protected by limited liability. In this paper, we investigate how the principal should construct the lottery to attain not only local but also global incentive compatibility. We first assess that the main issue with global incentive compatibility rests with intermediate types being potentially attractive reports to both lower and higher types. We then show that a lottery including three levels of profit (rather than only two) is optimal in that it is most likely to be globally incentive compatible under limited liability, if local incentive constraints are strictly satisfied. We identify conditions under which first best is implemented. In a setting with three types and three signals we also pin down the optimal distortions when those conditions are violated. We show that, if local incentive compatibility is not an issue but first best is beyond reach, then it is generally optimal to concede an information rent to one type only.

JEL-Codes: D820.

Keywords: informative signals, limited liability, conditional probability, incentive compatibility, full-rank condition.

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1 Introduction

Motivation

In principal-agent relationships, it is now well known how the principal can take advantage of some *ex-post* signal of the agent's private information to improve the efficiency of the trade. When the private information concerns the cost of running the activity, useful signals are the result of an audit, or the performance of another agent running a similar activity, which are naturally correlated with - hence, informative of - the cost. To take advantage of the signals, the principal offers a compensation scheme such that the agent is faced with a lottery, which associates a profit with each signal realization. The lower that the profit can be set for some signal realization(s), the more likely that the principal is to elicit information at no agency cost. Limited liability on the agent's side tightens the conditions under which this outcome is attained (Demougin and Garvie [3], Gary-Bobo and Spiegel [5]). On the other hand, those conditions also depend on the agent's cost function, which determines the benefit that the agent could derive by camouflaging information. In spite of a general recognition that the reliance on informative signals enhances contracting, hitherto no study has investigated for what technologies some contractual achievement can be attained, given the liability of the agent. The goal of our paper is to shed light on this issue.

To see why this is relevant, consider, for instance, an agent using the familiar Cobb-Douglas technology to produce some good or service for the principal. To fix ideas, take $q = \frac{1}{\theta} l^{\alpha} k^{\beta}$, where q is the level of output, l and k are the levels of two inputs (say, labour and capital), and θ is privately known (the type of the agent). With decreasing returns to scale ($\alpha + \beta < 1$), the cost is increasing and convex in the type, as is assumed by Gary-Bobo and Spiegel [5] (henceforth, GBS). With these technological features, high types are little motivated to mimic lower types, as they have much to loose in cost reimbursement in that case. The concern of the principal is thus to design lotteries in such a way as to extract information and retain surplus from the low types, which have much to gain in cost reimbursement, instead, if they over-report. By contrast, when returns to scale are increasing ($\alpha + \beta > 1$), as is likely, for instance, in regulated industries, the cost is increasing and concave in the type.[1] Then, not only is the principal concerned with ensuring that low types face sufficiently unattractive lotteries, through which any gain in cost reimbursement associated with type overstatement is washed out. She is also concerned with avoiding that high types face too attractive lotteries, which more than offset the penalty in cost reimbursement associated with type understatement. This is a complex issue in that two conflicting interests must be reconciled, as Riordan and Sappington [11] (henceforth RS) show. Indeed, the less attractive that the lotteries are made for low types, if they over-report, the more attractive that the lotteries become to high types, if they under-report. Whereas the two interests are handily harmonized when the cost of the agent is convex

[1] More precisely, regulated industries are often characterized by economies of scale, a concept that is related to but broader than that of increasing returns to scale.

in the type, no definitive conclusion can be drawn if the cost is concave instead, unless further analysis is developed. One would need to understand how to deal with the conflict, taking into account that limited liability on the agent's side imposes restrictions on the choice of the profits that can be assigned to each type.

As an additional example, one may consider the situation, already explored by the literature on incentive problems, in which the agent's cost is composed of a fixed cost and a privately known variable cost that are inversely related. This is the case, for instance, when the fixed cost is, in fact, an opportunity cost of renouncing to alternative businesses, which naturally increases with the efficiency of the agent in the trade with the principal. Obviously, depending on the shape of the two components, the total cost may take any shape with respect to the type (Maggi and Rodriguez-Clare [8]). In particular, the total cost is concave in the type when, as is highly plausible, the opportunity cost increases with the efficiency of the agent at a decreasing rate (Lewis and Sappington [7]). Then, the incentive issue pinpointed with regards to the Cobb-Douglas technology, comes out to be relevant again.

Setting and main results

We first consider a simple setting with three types and two or three informative signals. We look for conditions under which first best is implemented, as would be the case under complete information. With both two and three signals, we show that the principal deals with two issues, which have been looked at separately in the previous studies. First, as evidenced by GBS, the liability of the agent should be sufficiently high for the principal to be able to make the lotteries of the low types, if they over-report, sufficiently unattractive. That is, any gains in cost reimbursement associated with over-reporting is extracted through the lotteries. Second, as pointed out by RS, there should be no conflict between the incentive constraints whereby the extreme types are unwilling to mimic the intermediate type. This requires ensuring that the value of the lottery which the lowest type is faced with, if it over-reports, is not too negative. Indeed, as the value of that lottery gets lower, the value of the lottery that the highest type is faced with, if it under-reports, becomes higher and, hence, under-reporting becomes more attractive. As a result, it might be impossible to find a vector of profits for the intermediate type, which is unattractive to either of the extreme types.

Our main contribution is to show that, when there are three informative signals, rather than only two, the vectors of optimal profits may not be as characterized in the previous studies. In a setting with more than two signals, RS propose a compensation scheme such that each type of the agent is assigned an equal profit for all signal realizations but one (although profits differ across types, of course). They provide sufficient conditions for that compensation scheme to implement first best. GBS show that an efficient allocation is effected under limited liability only if it is effected with the compensation scheme proposed by RS. With three types, the intermediate type is assigned the same negative profit for all signal realizations but the one that the lowest type is least likely to draw. The principal can retain surplus from both the lowest and the intermediate type, only if that vector of profits satisfies the limited liability constraints. Any other incentive compatible vector of profits would tighten those constraints. We find that it might be necessary to opt for a different compensation scheme when the main difficulty for the principal is not that of preventing type overstatement but, rather, that of discouraging the extreme types from mimicking the intermediate type. Specifically, if it is possible to decrease one of the equal profits assigned to the intermediate type, given the liability of the agent, then the incentives to over-report of the lowest type are lessened. Whereas this makes under-reporting more attractive to the highest type, one can further fine-tune the remaining two profits of the intermediate type, in such a way as to avoid reinforcing the incentives of the highest type to understate. Overall, the essential benefit of this amended scheme is that it permits to effect an efficient allocation with a wider family of cost functions, including those that are (not too) concave in the type, which have been disregarded in agency problems with limited liability so far.

We next turn to consider a general setting with a continuum of types and a finite number of informative signals. The main result we derive is that, even if there are more than three signals that can be used in contractual design, the conditions that are necessary to effect an efficient allocation depend on the properties of three signals only. The reason is as in the three-type case. First, a low type is least motivated to overstate information if a high type receives more in the state which is least likely to be drawn by the low type than in any other state. Second, the conflict between upward and downward incentive constraints is weakest, if each type is faced with the highest sustainable deficit in all the remaining states but one. Third, the remaining profit is adjusted to attain local incentive compatibility. With these features, the compensation scheme is found to be an immediate extension of that pinned down in the three-type case, as obtained by setting the profit to the minimum for all the signal realizations other than the ones that are exploited to fine-tune incentives. As long as the necessary conditions hold, this compensation scheme is also sufficient for effecting an efficient allocation. Intuitively, provided that upward and downward incentive constraints are not in conflict, and that local incentive compatibility is attained, global incentive compatibility is at reach as well.

Once the optimal vector of profits is characterized for any given type, a cut-off level of liability is pinned down, which determines whether or not an efficient allocation can be effected, depending on the degree of concavity of the cost with respect to the type. The exact class of cost functions for which the efficient outcome is attained will naturally depend on the level of the agent's liability. Besides, it will also depend on the properties of the signal. In particular, the more informative that some signal is about the agent's type, the easier that it will be for the principal to induce information release and, hence, the more concave that the agent's cost is allowed to be with respect to the type.

As a final step, we study the contractual design in a case where the agent's liability is too low for the principal to be able to induce truthtelling without distortions and, hence, to effect an efficient allocation. To avoid technical complications that would not add to the economic insights, we develop this second-best analysis considering again three types and three informative signals. The analysis becomes quickly complex as it should be considered that, if a rent must be conceded to one extreme type to elicit information, then that type might become an attractive report to the other extreme type. However, looking at situations in which this issue does not arise, we show that the vectors of second-best profits displays the same structure as found in the first-best analysis. Particularly, this means that the intermediate type is faced three different levels of profits in the three states, rather than facing only two levels of profit as in the scheme of GBS. We further ascertain that the level of liability, which separates the regime in which local incentive compatibility is attained from that in which it is not, is also the level of liability which separates situations in which the optimal compensation includes three levels of profit from those in which it includes only two levels of profit. Noticeably, in the former situations, it is optimal to concede an information rent to only one of the extreme types, in general. Indeed, whereas it is impossible to find a vector of profits such that the intermediate type is an attractive report to none of the extreme types without giving up some surplus, any choice of profits that lessens the incentives to cheat of one such type ends up tightening the incentives to cheat of the other. Thus, the best for the principal is to pick, between the extreme types, the one to which conceding a rent is less costly. Intuitively, this is related to how likely the principal is to face one extreme type relative to the other.

1.1 Review of the literature

Our paper is related to the line of research pioneered by RS. They show that an efficient allocation is effected if the cost function is convex in the type whereas the probability function of the most informative signal is concave in the type. They also show that when there are only two informative signals, it is necessary and sufficient that the cost function be less concave in the type than the likelihood of the signal that is more informative between the two. In our results, the conditions on the cost function with respect to the type, which are both necessary and sufficient, depend not only on the properties of the likelihood function, as in RS, but also on the level of the agent's liability.

The first paper to consider limited liability on the agent's side in settings with correlated information is that of Demougin and Garvie [3]. In their model with a continuum of types and a binary signal, limited liability is represented in two alternative ways. First, the transfer from the principal to the agent cannot be negative, because the principal has no power to tax the agent under any conditions.[2] Second, the agent cannot incur any deficits, hence he will recover the entire cost of the activity, regardless of the signal realization. In line with GBS, we consider the latter kind of limited liability and assume that the agent can only be exposed to bounded deficits. Essentially, we refer to situations in which the principal is concerned with preserving the agent's financial viability, though not ensuring reimbursement of the entire cost. This is

[2] A similar form of limited liability is also represented in the two-type two-signal model of Kessler *et al.* [6], who allow for the transfer to be negative but not unbounded.

common practice, for instance, in regulated industries, where the firms' financial distress is generally prevented to avoid activity interruptions. Unlike in Demougin and Garvie [3] and Kessler *et al.* [6], and similarly to GBS, we allow for more than two informative signals, which is a crucial ingredient to our investigation. As already mentioned, unlike GBS, we also let the cost function take any shape with respect to the type.

As is well known, limited liability can alternatively be regarded as an extreme form of risk aversion. With that interpretation, our study is also related to Eso [4], who explores full surplus extraction in an agency problem with correlated information and risk aversion on the agent's side. Specifically, the author considers an auction in which the auctioneer/principal faces two potential buyers/agents, both risk averse. Their privately known valuations of the object offered for sale are correlated and can take only two values. By contrast, we develop the analysis considering a richer set of types. This extension enables us to capture the important circumstance that incentive compatibility is problematic essentially because intermediate types may potentially attract false reports by both lower types and higher types.

Lastly, our paper is related to the broader literature on mechanism design with multiple agents and correlated private information. Studies like Myerson [10], Crémer and McLean [2] (henceforth, CM) and McAfee and Reny [9], represent a seller/principal who offers an object through an auction, facing a number of potential buyers/agents whose privately known preferences for the object (their types) are correlated. In that environment, the signals of each agent's type are generated endogenously through the reports collected by the principal from the other agents. In our setting, as in RS and GBS, the signals are exogenous, instead. However, regardless of whether correlated information is endogenous or exogenous, similar results are obtained on first-best implementation. Indeed, the necessary and sufficient conditions, which are shown to be valid for *any utility function* of the agents in the studies aforementioned, would carry over for any cost function of the agent in frameworks like ours. Our analysis pins down the properties that the cost function should display for that outcome to be attained under limited liability.

1.2 Outline

The reminder of the article is organized as follows. In section 2 we describe the model. In sections 3 and 4 we develop the analysis of first-best implementation. We first consider a discrete number of types and then allow for a continuum of types. In section 5 we return to a discrete-type framework to investigate the optimal second-best contracting in a case where first-best implementation is beyond reach because the level of liability is too low. We conclude in section 6. Mathematical proofs are relegated to an appendix.

2 The model

A principal P delegates the production of q units of a good (or service) to an agent. They are both risk neutral. The agent incurs the cost of production $C(q,\theta)$, where $\theta \in \Theta \equiv [\underline{\theta}, \overline{\theta}]$ parametrizes his productivity. As a compensation, he receives a payment of z from P, who derives the gross utility $S(q)$ from consumption. The cost function $C(\cdot,\cdot)$ is twice continuously differentiable in either argument, with the following partial and cross derivatives: $C_q(q,\theta) > 0$, $C_\theta(q,\theta) > 0$ and $C_{q\theta}(q,\theta) > 0$. In words, the total cost increases in the size of the production; both the total cost and the marginal cost increase with θ. We impose no restriction on the sign of the second derivative $C_{\theta\theta}(q,\theta)$. We only require the sign to be the same for all values of q and θ. The function $S(\cdot)$ is twice continuously differentiable with derivatives $S'(\cdot) > 0$ and $S''(\cdot) < 0$. Moreover, $S(0) = 0$ and the Inada's conditions are satisfied.

In the contracting stage, the productivity parameter θ is privately known to the agent and represents his type. It is commonly known that the distribution function of θ is $F(\theta)$ with $F'(\theta) \equiv f(\theta)$. The agent's type is correlated with a random signal s, which is observed publicly after the contract is drawn up and the output is produced (or chosen) and before the compensation is made. The realized signal (the "state of nature") is hard information, involving that a legally enforceable contract can be signed upon.[3] We take the signal to be drawn from the discrete support $N \equiv \{1,..,n\}$, where $n \geq 2$. The probability that signal s is realized conditional on the type being θ is $p_s(\theta)$. We assume that $p_s(\theta) > 0$, $\forall s$, $\forall\theta$, and that the monotonic likelihood property, which is standard in mechanism design, is here satisfied in a weak sense:

$$\frac{p_1(\theta)}{p_1(\theta')} > \frac{p_s(\theta)}{p_s(\theta')} > \frac{p_n(\theta)}{p_n(\theta')},\ \forall\theta > \theta',\ \forall s \neq 1, n,\ \text{if } n > 2. \tag{1}$$

With this property in place, we shall say that there are at least three "informative" signals. Once (1) is reformulated as

$$\frac{p_1'(\theta)}{p_1(\theta)} > \frac{p_s'(\theta)}{p_s(\theta)} > \frac{p_n'(\theta)}{p_n(\theta)},\ \forall s \neq 1, n,\ \forall\theta,$$

saying that a signal is informative of the agent's type is tantamount to saying that the rate of change of its likelihood with respect to the type differs from that of any other signal. Formally, the above triplet of conditions shows that, for any subset $N' \subset N$ containing signal 1 and some other signal $s \neq 1$, the rate of change of 1 is higher than that of s; for any subset $N'' \subset N$ containing signal n and some other signal $s \neq n$, the rate of change of n is lower than that of s.

Invoking the Revelation Principle, we can confine attention to contractual offers $\{q(\theta), \mathbf{z}(\theta)\}$, $\forall\theta$, in which $q(\theta)$ is the quantity an agent of type θ is required to produce and $\mathbf{z}(\theta) \equiv$

[3] For instance, in regulatory settings, the agent is a regulated firm and the signal can be the behaviour or the market performance of another firm, operating either in the same (or a similar) sector, which conveys information about the cost of the regulated firm. In other contexts, the signal can be the outcome of an audit of the activity run by the agent.

$(z_1(\theta)), ..., z_n(\theta))$ is the vector of the transfers he is assigned in the different states. The net surplus of P in state s is $S(q(\theta)) - z_s(\theta)$. Denote $\widetilde{\pi}_s(\theta' | \theta) \equiv z_s(\theta') - C(q(\theta'), \theta)$ the profit an agent of type θ obtains in state s when he announces θ' to P. Knowing that $z_s(\theta') = \pi_s(\theta') + C(q(\theta'), \theta')$, we can rewrite

$$\widetilde{\pi}_s(\theta' | \theta) = C(q(\theta'), \theta') - C(q(\theta'), \theta) + \pi_s(\theta'), \tag{2}$$

This reduces to $\widetilde{\pi}_s(\theta | \theta) = \pi_s(\theta)$, if the agent reveals his type. The vector of profits he faces in that case is thus given by $\boldsymbol{\pi}(\theta) \equiv (\pi_1(\theta), ..., \pi_n(\theta))$. We will say that, when he tells the truth and signal s is realized, an agent of type θ receives a *reward* if $\pi_s(\theta) > 0$ and incurs a *punishment* if $\pi_s(\theta) < 0$.[4] Consistent with this, the programme of P, to be presented in the next subsection, depends only on the profits rather than on the exact structure of the transfers assigned to the various types in the different states.

The relationship between P and the agent unfolds as follows. Before contracting takes place, nature draws θ and the agent learns its realization. P addresses the contractual offer to the agent. If the agent rejects the offer, then the parties obtain their reservation payoffs and the relationship ends. If the agent accepts the offer, then he makes a report about his type to P (or, alternatively, he picks an option within the contractual menu) and produces accordingly. Next, the signal is realized and the contractually specified transfer is paid.

2.1 The programme of the principal

For any given type θ and signal s, the *ex-post* net surplus of P is reformulated as $S(q(\theta)) - C(q(\theta), \theta) - \pi_s(\theta)$, provided that the agent reveals his type in equilibrium. Accordingly, referring to the profit $\pi_s(\theta)$ rather than to the transfer $z_s(\theta)$, with a standard change of variable, and letting $R(\theta) \equiv \sum_{s=1}^{n} \pi_s(\theta) p_s(\theta)$ denote the expected profit of the agent, the programme of P, to be denoted Γ, is formulated as follows:[5]

$$\underset{\{q(\theta);\boldsymbol{\pi}(\theta),\forall\theta\}}{Max} \int_{\theta\in\Theta} (S(q(\theta)) - C(q(\theta), \theta) - R(\theta)) f(\theta)$$

subject to

$$R(\theta) \geq C(q(\theta'), \theta') - C(q(\theta'), \theta) + \sum_{s=1}^{n} \pi_s(\theta') p_s(\theta), \ \forall\theta, \theta' \tag{IC}$$

$$R(\theta) \geq 0, \ \forall\theta \tag{PC}$$

$$\pi_s(\theta) \geq -L, \ \forall\theta, \ \forall s. \tag{LL}$$

[4]It is useful to remark that (2) would be the same if $z_s(\theta)$ were to include a fixed component related to the type and a stochastic component conditional on the signal realization, as considered by Bose and Zhao [1].

[5]A more general formulation of the objective function in the programme of P would be $\int_\theta (S(q(\theta)) - C(q(\theta), \theta) - (1-\omega) R(\theta)) f(\theta)$, for some $\omega \in (0, 1)$. With this formulation, results would be qualitatively the same in that P dislikes giving a rent to the agent and would attempt to set profits such that $R(\theta)$ is as low as possible. Accounting for this, we simply take $\omega = 0$.

(IC) is the incentive compatibility constraint whereby an agent of type θ prefers revealing his true type rather than cheating. (PC) is the participation constraint which ensures that his expected profit is non-negative. (LL) is the limited liability constraint which ensures that the maximum deficit to which the agent is exposed, does not exceed $L > 0$ in each possible state. Essentially, this form of limited liability represents situations in which the principal would like to avoid the agent becoming so financially distressed that the activity must be interrupted, at least as long as the agent does not attempt to conceal information.[6] Overall, the programme is standard, except that the cost of production can take any shape with respect to the type θ.

The first part of our study will be devoted to investigating under what conditions and in which way P effects the first-best allocation. This is defined by the optimality condition $S'(q^{fb}(\theta)) = C_q\left(q^{fb}(\theta), \theta\right), \forall\theta$, together with the surplus extraction constraint

$$R(\theta) = 0, \ \forall\theta. \tag{3}$$

In what follows, unless differently specified, we will use the notation

$$\Delta C(\theta, \theta') \equiv C\left(q^{fb}(\theta), \theta\right) - C\left(q^{fb}(\theta), \theta'\right), \ \forall (\theta', \theta).$$

Also, we will let $\Pi(\theta)$ be the set of vectors $\pi(\theta)$, the elements of which satisfy (PC) as an equality (so that (3) holds) and, in addition, satisfy (IC). We will have to verify that $\mathbf{\Pi}(\theta)$ is non-empty, *i.e.*, that the efficient allocation is incentive compatible. Besides, we will have to verify that there exist vectors in $\mathbf{\Pi}(\theta)$ that satisfy (LL) as well.

2.2 Previous findings

Before turning to the analysis, it is useful to recall the previous findings on first-best implementation in settings with informative signals which are relevant for our study.

RS Assume that $C(\cdot, \theta)$ is strictly convex in the type θ, and that $\exists i \in N$ such that $p_i(\theta)$ is increasing and concave in θ. Taking the type to be drawn from the discrete set $\{\theta_1, \theta_2, ..., \theta_T\}$, instead of Θ, so that the generic value specifies as θ_t, and also taking $L \to \infty$, $\mathbf{\Pi}(\theta_t)$ is non-empty for any θ_t. After presenting this result in Corollary 1.4, RS show that the principal effects the first-best allocation by adopting the binary vector of profits $\boldsymbol{\pi}^i(\theta_t) \in \mathbf{\Pi}(\theta_t)$, which

[6]For instance, in regulated industries, in which this is common practice, L could be interpreted as an indicator of financial viability, beyond which the regulated firm would go bankrupt. Remark that the limited liability constraints are required to hold as long as the agent does not conceal information. This involves that, even if the profits are set such that the agent does not lose more than L in equilibrium, he might still incur a greater loss, should he decide to deliver an out-of-equilibrium report.

is defined for any $t \geq 2$ as

$$\pi_i(\theta_t) = \Delta C(\theta_t, \theta_{t-1}) \frac{1 - p_i(\theta_t)}{p_i(\theta_t) - p_i(\theta_{t-1})} \quad (4)$$

$$\pi_s(\theta_t) = -\Delta C(\theta_t, \theta_{t-1}) \frac{p_i(\theta_t)}{p_i(\theta_t) - p_i(\theta_{t-1})}, \ \forall s \neq i, \quad (5)$$

In Corollary 1.5, RS further show that, if $n = 2$, then $\boldsymbol{\pi}^i(\theta_t) \in \mathbf{\Pi}(\theta_t)$ if and only if

$$\frac{\Delta C(\theta_2, \theta_1)}{-\Delta C(\theta_2, \theta_3)} \leq \frac{p_i(\theta_2) - p_i(\theta_1)}{p_i(\theta_3) - p_i(\theta_2)}. \quad (6)$$

This is ensured if the cost function is less concave in type than the conditional probability of signal i. In general, (6) can be satisfied when types θ_1 and θ_2 have similar costs of producing output q, relative to types θ_2 and θ_3, and/or when types θ_2 and θ_3 have similar probabilities of drawing signal i, relative to types θ_1 and θ_2.

GBS Assume that $p_s(\theta)$ is continuously differentiable and that $\exists i$ such that $i = \arg\max_{s \in N} \left\{ \frac{p_s'(\theta)}{p_s(\theta)} \right\}$, $\forall \theta$. Then, under the conditions identified by RS, it is found that $\pi^i(\theta) \in \Pi(\theta)$, where $\pi^i(\theta)$ is now (re)defined as

$$\pi_i(\theta) = C_\theta\left(q^{fb}(\theta), \theta\right) \frac{1 - p_i(\theta)}{p_i'(\theta)} \quad (7)$$

$$\pi_s(\theta) = -C_\theta\left(q^{fb}(\theta), \theta\right) \frac{p_i(\theta)}{p_i'(\theta)}, \ \forall s \neq i. \quad (8)$$

Moreover, among all the elements of $\Pi(\theta)$, $\boldsymbol{\pi}^i(\theta)$ is most likely to satisfy (LL). Being based on (8), we see that (LL) holds if and only if

$$C_\theta\left(q^{fb}(\theta), \theta\right) \frac{p_i(\theta)}{p_i'(\theta)} \leq L, \ \forall \theta. \quad (9)$$

CM Take $L \to \infty$ and, in line with RS, let the type θ_t be drawn from the discrete set $\{\theta_1, \theta_2, ..., \theta_T\}$. As long as the vectors $\mathbf{p}(\theta_t) \equiv \{p_1(\theta_t), ..., p_n(\theta_t)\}$ are linearly independent across types, $\Pi(\theta_t)$ is non-empty for all θ_t. This follows from Farkas' lemma, which implies that for all θ_t there exists a $n-$dimensional vector $\mathbf{h}(\theta_t) \equiv \{h_1(\theta_t), ..., h_n(\theta_t)\}$ such that the following two conditions hold:

$$\sum_{s=1}^{n} h_s(\theta_t) p_s(\theta_t) = 0, \ \forall \theta_t \in \Theta_T \quad (10)$$

$$\sum_{s=1}^{n} h_s(\theta_t) p_s(\theta_{t'}) < 0, \ \forall \theta_t, \theta_{t'} \in \Theta_T. \quad (11)$$

By setting $\pi_s(\theta_t) = \gamma_t h_s(\theta_t)$, $\forall s$, $\forall t$, and choosing the "scaling" parameter γ_t arbitrarily big, all surplus is extracted from type θ_t and no incentive to mimic θ_t is triggered for any other type. First best is beyond reach if there exists some type θ_t for which no vector $\mathbf{h}(\theta_t)$ can be found such that (10) and (11) are satisfied.[7]

In substance, RS highlight that, as long as the agent can be imposed unlimited punishments, first best is possibly at hand even when the set of informative signals includes only two elements. As is evident from the definition of $\boldsymbol{\pi}^i(\theta_t)$, the agent's gain only depends on whether signal i is realized, rather than any other signal, regardless of how rich the subset of other signals is. From GBS we further retain that any other vector of profits belonging to $\Pi(\theta_t)$ includes an element, the value of which is below that of (5), involving that it is more difficult to satisfy (LL). Given property (1), $i = 1$ in our framework. The best known result in agency problems with correlated information is perhaps that of CM, who show that the first-best outcome is attained if the vectors of conditional probabilities of the signals are linearly independent. Importantly, this result is obtained regardless of the properties of the cost function. By setting rewards and punishments arbitrarily high, any untruthful report can be made unattractive. However, high punishments are unfeasible when the agent is protected by limited liability. One then needs to consider the properties of the cost and the probability functions to ascertain whether there exists some vector of profits that permits first-best implementation under limited liability, consistent with the analysis developed by GBS.

Our goal is to extend the analysis beyond that of GBS and investigate whether first best is at reach when (6) and (9) are not jointly satisfied, and what vector of profits should be adopted in that case. Indeed, with property (1) being verified, (9) is relaxed to the utmost for signal $i = 1$. Therefore, one will naturally think that the vector $\boldsymbol{\pi}^1(\theta_t)$ should be used. However, for $\boldsymbol{\pi}^1(\theta_t)$ to be incentive compatible, it must be such that (6) holds, as is known from RS. This might not be the case though. Whereas the assumption that some signal displays the highest likelihood ratio for all types is similar to that introduced by GBS, the assumption that some other signal displays the lowest likelihood ratio, embodied in property (1), is made for the purpose of our study. We show in Appendix that property (1) entails that the full-rank condition of CM must be satisfied for the extreme types but not necessarily for the other types.[8] In this respect, our analysis diverges from that of CM and comes closer to that of RS and GBS.

[7] The "only if" proof of CM shows that if the vector $\mathbf{h}(\theta_T)$ does not exist, then it is impossible to ensure that θ_T is not an attractive report to any type $\theta_t < \theta_T$. Notice however that the existence of $\mathbf{h}(\theta_t)$ is not necessary for all types. In particular, it does not need to hold for type θ_1. This paves the way for the results drawn in the study of RS, in which first-best implementation does not necessarily depend on the existence of $\mathbf{h}(\theta_t)$, $\forall t$. Bose and Zhao [1] show that Proposition 1 in RS implies that first best might be effected when the full-rank condition is violated.

[8] In Appendix A we consider a discrete type set, in line with RS and CM. We show that, as long as property (1) holds, $\mathbf{p}(\theta_1)$ and $\mathbf{p}(\theta_T)$ do not lie in the convex hull generated by the probability vectors of the other types. Moreover, there exist vectors $\mathbf{p}(\theta_t)$, $t \neq 1, T$, which lie in the convex hull generated by the probability vectors of the other types and do not violate conditions (1).

3 Three types and two or three signals

We begin by considering a simple setting with three types. We will highlight the characteristics that the vectors of profits should display when there are two and three signals.

3.1 Two signals

For the time being, we focus on the discrete set of types $\{\theta_1, \theta_2, \theta_3\}$ and take $n = 2$. Property (1), which refers to a case where $n \geq 3$, is replaced by

$$\frac{p_1(\theta_t)}{p_1(\theta_{t'})} > \frac{p_2(\theta_t)}{p_2(\theta_{t'})},\ \forall \theta_t > \theta_{t'}. \tag{12}$$

Consider any binary vector of profits $(\pi_1(\theta_t), \pi_2(\theta_t))$ belonging to $\mathbf{\Pi}(\theta_t)$, designed for the generic type $\theta_t \in \Theta$, such that $\pi_2(\theta_t) < 0 < \pi_1(\theta_t)$. Given that surplus extraction requires (3) to hold, we can express $\pi_1(\theta_t)$ in terms of $\pi_2(\theta_t)$ as $\pi_1(\theta_t) = -\pi_2(\theta_t) p_2(\theta_t) / p_1(\theta_t)$. This expression is useful to formulate the expected value of the lottery which type $\theta_{t'}$ is faced with, if it announces θ_t, in terms of $\pi_2(\theta_t)$ only. Specifically, that lottery grants $-\pi_2(\theta_t) p_2(\theta_t) / p_1(\theta_t)$ with probability $p_1(\theta_{t'})$, and $\pi_2(\theta_t)$ with probability $p_2(\theta_{t'})$, so that its expected value to type $\theta_{t'}$ is $\pi_2(\theta_t) p_2(\theta_t) \left(\frac{p_2(\theta_{t'})}{p_2(\theta_t)} - \frac{p_1(\theta_{t'})}{p_1(\theta_t)} \right)$. Because $\pi_2(\theta_t) < 0$, given property (12), that value is negative if $\theta_{t'} < \theta_t$, and positive in the converse case. That is, the profits designed for type θ_t penalizes a lower type $\theta_{t'}$, if it announces θ_t, because, as compared to θ_t, type $\theta_{t'}$ is more likely to draw signal 2 and less likely to draw signal 1. Conversely, those profits favour a higher type $\theta_{t'}$, if it announces θ_t, because, as compared to θ_t, type $\theta_{t'}$ is now less likely to draw signal 2 and more likely to draw signal 1. In addition to the lottery, the payoff of type $\theta_{t'}$, if it announces θ_t, includes the difference between the (false) cost reimbursed by P and the (real) cost incurred to perform the task. Overall, the payoff of type $\theta_{t'}$, if it announces θ_t, is given by

$$\pi_2(\theta_t) p_2(\theta_t) \left(\frac{p_2(\theta_{t'})}{p_2(\theta_t)} - \frac{p_1(\theta_{t'})}{p_1(\theta_t)} \right) + \Delta C(\theta_t, \theta_{t'}).$$

The above expression is suggestive of what may incentivize type $\theta_{t'}$ to report θ_t. If $\theta_t > \theta_{t'}$, then type $\theta_{t'}$ loses in terms of lottery by reporting θ_t, but gains in terms of cost reimbursement (since $\Delta C(\theta_t, \theta_{t'}) > 0$ in that case). On the opposite, if $\theta_t < \theta_{t'}$, then type $\theta_{t'}$ loses in terms of cost reimbursement (since $\Delta C(\theta_t, \theta_{t'}) < 0$) but gains in terms of lottery. Thus, for both lower and higher types, there are two opposite effects at work. Remark that such effects follow from type θ_t being rewarded in state 1 and punished in state 2. In the converse case, given property (12), lower types would obviously want to announce θ_t because, by doing so, they would gain in terms of both lottery and cost reimbursement, which justifies our choice to consider a pair of profits such that $\pi_2(\theta_t) < 0 < \pi_1(\theta_t)$ in the first place. Taking this all into account, one can assess what requirements the profits should verify for not attracting false reports. In particular, the profits of the three types in state 2 must be respectively such that

$$\pi_2(\theta_1) \geq \frac{\Delta C(\theta_1, \theta_{t'})}{p_2(\theta_1)\left(\frac{p_1(\theta_{t'})}{p_1(\theta_1)} - \frac{p_2(\theta_{t'})}{p_2(\theta_1)}\right)}, \ t' = 2, 3, \tag{13}$$

$$\frac{\Delta C(\theta_2, \theta_3)}{p_2(\theta_2)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_2(\theta_3)}{p_2(\theta_2)}\right)} \leq \pi_2(\theta_2) \leq \frac{-\Delta C(\theta_2, \theta_1)}{p_2(\theta_2)\left(\frac{p_2(\theta_1)}{p_2(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}\right)} \tag{14}$$

and

$$\pi_2(\theta_3) \leq \frac{-\Delta C(\theta_3, \theta_{t'})}{p_2(\theta_3)\left(\frac{p_2(\theta_{t'})}{p_2(\theta_3)} - \frac{p_1(\theta_{t'})}{p_1(\theta_3)}\right)}, \ t' = 1, 2. \tag{15}$$

Considering these requirements together with limited liability, two kinds of necessary conditions are identified. There are, first, the conditions due to limited liability, which represent the counterpart of (9) - the condition established by GBS - in a setting with three types and two signals:

$$\Delta C(\theta_2, \theta_1) \frac{p_1(\theta_2)}{p_1(\theta_2) - p_1(\theta_1)} \leq L, \tag{16}$$

$$\Delta C(\theta_3, \theta_{t'}) \frac{p_1(\theta_3)}{p_1(\theta_3) - p_1(\theta_{t'})} \leq L, \ \forall t' \in \{1, 2\}. \tag{17}$$

Besides, there is one more condition, which follows from (14):

$$\frac{\Delta C(\theta_2, \theta_1)}{-\Delta C(\theta_2, \theta_3)} \leq \frac{\frac{p_1(\theta_2) - p_1(\theta_1)}{p_1(\theta_2)} - \frac{p_2(\theta_2) - p_2(\theta_1)}{p_2(\theta_2)}}{\frac{p_1(\theta_3) - p_1(\theta_2)}{p_1(\theta_2)} - \frac{p_2(\theta_3) - p_2(\theta_2)}{p_2(\theta_2)}}. \tag{18}$$

With a simple manipulation in which the property $p_2(\cdot) = 1 - p_1(\cdot)$ is used, (18) is rewritten as (6) - the condition established by RS.

Proposition 1 *Take* $\theta \in \{\theta_1, \theta_2, \theta_3\}$ *and* $n = 2$. $\mathbf{\Pi}(\theta_t)$ *is non-empty* $\forall t \in \{1, 3\}$.

(*GBS*) *Assume that* $\mathbf{\Pi}(\theta_2)$ *is non-empty.* $\exists \boldsymbol{\pi}(\theta_t) \in \mathbf{\Pi}(\theta_t)$ *such that* (*LL*) *holds* $\forall t$ *if and only if* (16) *and* (17) *are satisfied.*

(*RS*) $\mathbf{\Pi}(\theta_2)$ *is non-empty if and only if* (6) *is satisfied.*

For further analysis, it is useful to refer to (18), rather than to its equivalent formulation in (6). Let us interpret (18). Given that type θ_1 gains in cost reimbursement and loses in terms of lottery, if it claims θ_2, whereas the converse occurs for type θ_3, there exist values of $\pi_2(\theta_2)$ such that types θ_1 and θ_3 are both discouraged from claiming θ_2 if and only if the ratio between the gain to type θ_1 and the loss to type θ_3 in terms of cost reimbursement does not exceed the ratio between the loss to type θ_1 and the gain to type θ_3 in terms of lottery. In line with the interpretation of (6), the gain/loss ratio in terms of cost reimbursement does not exceed the loss/gain ratio in terms of lottery if and only if the cost is less concave (more convex) than the conditional probability of signal 1.

Obviously, the conditions stated in Proposition 1 are also sufficient for first-best implementation. Indeed, if they are satisfied, then there exists a vector $\boldsymbol{\pi}(\theta_t) \in \boldsymbol{\Pi}(\theta_t)$, $\forall \theta_t$, such that (LL) holds. In definitive, when the signal is binary and there are only three types, the conditions for first-best implementation are confirmed to be as established by the previous literature.

Corollary 1 *Take $\theta \in \{\theta_1, \theta_2, \theta_3\}$ and $n = 2$. Provided that (18) to (17) are satisfied, first best is implemented if the vectors of profits are as follows:*

$$\begin{aligned}
\boldsymbol{\pi}^1(\theta_1) &= (0,0) \\
\boldsymbol{\pi}^1(\theta_2) &= \left(\Delta C(\theta_2,\theta_1)\frac{1-p_1(\theta_2)}{p_1(\theta_2)-p_1(\theta_1)}, -\Delta C(\theta_2,\theta_1)\frac{p_1(\theta_2)}{p_1(\theta_2)-p_1(\theta_1)}\right) \\
\boldsymbol{\pi}^1(\theta_3) &= \left(L\frac{1-p_1(\theta_3)}{p_1(\theta_3)}, -L\right).
\end{aligned}$$

Let us first discuss the design of the profits of the extreme types. Being based on (13) and (15), it is immediate to see that it is not an issue to find profits such that neither θ_1 nor θ_3 is an attractive report to any other type, as long as the necessary conditions hold. First, it suffices to set $\pi_s(\theta_1) = 0$, $\forall s$, because, in that case, higher types lose money, if they produce $q^{fb}(\theta_1)$ and are reimbursed $C\left(q^{fb}(\theta_1), \theta_1\right)$ rather than their true cost. Second, types lower than θ_3 are least motivated to claim θ_3, if this type is assigned the lowest possible profit $(-L)$ in state 2, which lower types are more likely to draw. Thus, P can set $\pi_2(\theta_3) = -L$ and, accordingly, $\pi_1(\theta_3) = L(1 - p_1(\theta_3))/p_1(\theta_3)$. As far as the intermediate type is concerned, once $\pi_2(\theta_2)$ is set to comply with (14), one can use (3) to determine $\pi_1(\theta_2)$. In particular, setting $\pi_2(\theta_2)$ to the higher between the upper bound to the range of feasible values identified in (14) and $-L$, namely $\pi_2(\theta_2) = -\Delta C(\theta_2, \theta_1)\frac{p_1(\theta_2)}{p_1(\theta_2)-p_1(\theta_1)}$, then $\pi_1(\theta_2)$ and the vector $\boldsymbol{\pi}^1(\theta_2)$ are found accordingly. The profits in $\boldsymbol{\pi}^1(\theta_2)$ are just the profits defined in (4) and (5) for the case of a binary signal and $\theta_t = \theta_2$.

3.2 Three signals

We found that when the signal is binary, the best compensation scheme that P can use to effect the first-best allocation is the scheme first proposed by RS and then reconsidered by GBS in a setting with a continuum of types and limited liability. We now look at a case where $n = 3$. This extension will allow us to show that when there is an additional signal, it might be optimal to offer the intermediate type a different profit for each signal realization. Proceeding as above, we formulate the incentive constraints whereby θ_t is an attractive report neither to lower types nor to higher types as follows:

$$\pi_3(\theta_t) \leq \frac{\Delta C(\theta_t,\theta_{t'}) + \pi_2(\theta_t)\, p_2(\theta_t)\left(\frac{p_2(\theta_{t'})}{p_2(\theta_t)} - \frac{p_1(\theta_{t'})}{p_1(\theta_t)}\right)}{-p_3(\theta_t)\left(\frac{p_3(\theta_{t'})}{p_3(\theta_t)} - \frac{p_1(\theta_{t'})}{p_1(\theta_t)}\right)},\ \forall\theta_{t'} < \theta_t \tag{19}$$

$$\pi_3(\theta_t) \geq \frac{\Delta C(\theta_t,\theta_{t'}) - \pi_2(\theta_t)\, p_2(\theta_t)\left(\frac{p_1(\theta_{t'})}{p_1(\theta_t)} - \frac{p_2(\theta_{t'})}{p_2(\theta_t)}\right)}{p_3(\theta_t)\left(\frac{p_1(\theta_{t'})}{p_1(\theta_t)} - \frac{p_3(\theta_{t'})}{p_3(\theta_t)}\right)},\ \forall\theta_{t'} > \theta_t. \tag{20}$$

These two conditions, which are the counterpart of (13) - (15) in situations with three signals, are expressed in terms of two profits, rather than only one as in the case of a binary signal. It is easy to verify that also in this setting, just as with a binary signal, there are two kinds of necessary conditions to be verified. First, the incentive constraints should not be in conflict with the limited liability constraints, namely (19) should hold together with $\pi_3(\theta_t) \geq -L$. Second, there should be no conflict between the incentive constraints whereby the extreme types are unwilling to mimic the intermediate type.

Let us begin with the first potential conflict. It turns out that the additional signal does not relax the conditions under which it is eliminated. Indeed, (19) is not in conflict with the limited liability constraint in state 3 if and only if it is not when $\pi_3(\theta_t) = \pi_2(\theta_t) < 0$. To see it, start from this equality and consider raising one of the two profits, say, $\pi_3(\theta_t)$. Then, at least one of $\pi_1(\theta_t)$ and $\pi_2(\theta_t)$ must be decreased. To avoid tightening the limited liability constraints, one should reduce $\pi_1(\theta_t)$ instead of $\pi_2(\theta_t)$. However, associated with the raise in $\pi_3(\theta_t)$, a reduction in $\pi_1(\theta_t)$ would strengthen the incentives of some lower type to report θ_t more than would a reduction in $\pi_2(\theta_t)$, provided lower types are less likely to draw signal 1 than signal 2. It follows that (16) and (17) are necessary also in a case where $n = 3$.

Let us now consider the second potential conflict. The possibility of using an additional signal induces a critical change in this respect. Indeed, the necessary condition (18) is replaced by

$$\frac{\Delta C(\theta_2,\theta_1)}{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} - \frac{-\Delta C(\theta_2,\theta_3)}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}} \leq -\pi_2(\theta_2)\, p_2(\theta_2)\left(\psi_{2,3}(\theta_1,\theta_2) - \psi_{2,3}(\theta_3,\theta_2)\right), \tag{21}$$

where the following definition has been used:

$$\psi_{s,n}(\theta',\theta) \equiv \frac{\frac{p_1(\theta')}{p_1(\theta)} - \frac{p_s(\theta')}{p_s(\theta)}}{\frac{p_1(\theta')}{p_1(\theta)} - \frac{p_n(\theta')}{p_n(\theta)}}.$$

If $\psi_{2,3}(\theta_1,\theta_2) > \psi_{2,3}(\theta_3,\theta_2)$, then (21) is weakest when $\pi_2(\theta_2)$ is set to $-L$. To interpret, first rewrite the inequality as

$$\frac{\frac{p_1(\theta_2)-p_1(\theta_1)}{p_1(\theta_2)} - \frac{p_2(\theta_2)-p_2(\theta_1)}{p_2(\theta_2)}}{\frac{p_1(\theta_3)-p_1(\theta_2)}{p_1(\theta_2)} - \frac{p_2(\theta_3)-p_2(\theta_2)}{p_2(\theta_2)}} > \frac{\frac{p_1(\theta_2)-p_1(\theta_1)}{p_1(\theta_2)} - \frac{p_3(\theta_2)-p_3(\theta_1)}{p_3(\theta_2)}}{\frac{p_1(\theta_3)-p_1(\theta_2)}{p_1(\theta_2)} - \frac{p_3(\theta_3)-p_3(\theta_2)}{p_3(\theta_2)}} \tag{22}$$

and observe that the left-hand side of (22) replicates the right-hand side of (18), and that the right-hand side of (22) is just the same as the left-hand side, except that the likelihood of signal 3 replaces the likelihood of signal 2. Next consider that, if $\pi_2(\theta_2)$ is decreased, then by announcing θ_2 instead of telling the truth, type θ_1 loses and type θ_3 gains in terms of lottery. Having (22) satisfied means that the ratio between such loss and gain exceeds the ratio between the loss to type θ_1 and the gain to type θ_3 that would result from a decrease in $\pi_3(\theta_2)$ (instead of $\pi_2(\theta_2)$). Then, the best is to set $\pi_2(\theta_2) = -L$. Obviously, in the converse case, the best would be to set $\pi_3(\theta_2) = -L$, instead. For simplicity, we refer to the case where (22) is satisfied indeed, namely $\psi_{2,3}(\theta_1, \theta_2) > \psi_{2,3}(\theta_3, \theta_2)$, and it is optimal to set $\pi_2(\theta_2) = -L$. This leads us to derive our next result. As in Proposition 1, we identify conditions for the two potential conflicts not to arise. To that end, it is useful to introduce the following notation:

$$\Psi_{1,n}(\theta', \theta, \theta'') = \frac{\frac{\Delta C(\theta,\theta')}{\frac{p_n(\theta')}{p_n(\theta)} - \frac{p_1(\theta')}{p_1(\theta)}} - \frac{-\Delta C(\theta,\theta'')}{\frac{p_1(\theta'')}{p_1(\theta)} - \frac{p_n(\theta'')}{p_n(\theta)}}}{\sum_{s\neq 1,n} p_s(\theta)\left(\psi_{s,n}(\theta', \theta) - \psi_{s,n}(\theta'', \theta)\right)}.$$

Proposition 2 *Take $\theta \in \{\theta_1, \theta_2, \theta_3\}$ and $n = 3$. $\mathbf{\Pi}(\theta_t)$ is non-empty $\forall t \in \{1, 3\}$.*

Assume that $\mathbf{\Pi}(\theta_2)$ is non-empty. $\exists \boldsymbol{\pi}(\theta_t) \in \mathbf{\Pi}(\theta_t)$ such that (LL) holds $\forall t$ if and only if (16) *and* (17) *are satisfied.*

Assume that $\psi_{2,3}(\theta_1, \theta_2) > \psi_{2,3}(\theta_3, \theta_2)$. $\mathbf{\Pi}(\theta_2)$ is non-empty if and only if

$$L \geq \Psi_{1,3}(\theta_1, \theta_2, \theta_3). \tag{23}$$

According to the proposition, first-best implementation rests critically on (23). With this condition satisfied, P can find vectors of profits such that the incentives to lie upwards are eliminated jointly with the incentives to lie downwards and, hence, incentive compatibility is attained in any reporting directions. This condition establishes a clear relationship between the properties of the cost function with respect to the type, the likelihood function, and the level of liability. In particular, depending on the agent's liability, (23) is satisfied for cost functions that are not too concave with respect to the type. The results in Proposition 2 are obtained with the compensation scheme that we now present.

Corollary 2 *Take $\theta \in \{\theta_1, \theta_2, \theta_3\}$ and $n = 3$. Under* (16), (17) *and* (23), *first best is implemented if the vectors of profits are as follows:*

$$\begin{aligned}
\boldsymbol{\pi}^*(\theta_1) &= (0,0,0) \\
\boldsymbol{\pi}^*(\theta_2) &= \left(\frac{\Delta C(\theta_2,\theta_1) - L\frac{p_3(\theta_2)-p_3(\theta_1)}{p_3(\theta_2)}}{p_1(\theta_2)\left(\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}\right)} - L, -L, \frac{\Delta C(\theta_2,\theta_1) - L\frac{p_1(\theta_2)-p_1(\theta_1)}{p_1(\theta_2)}}{-p_3(\theta_2)\left(\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}\right)} - L\right) \\
\boldsymbol{\pi}^*(\theta_3) &= \left(L\frac{1-p_1(\theta_3)}{p_1(\theta_3)}, -L, -L\right),
\end{aligned}$$

where $\pi_3^*(\theta_2) > \pi_2^*(\theta_2) = -L$.

If (18) *is violated, then* $\exists \boldsymbol{\pi}(\theta_2) \in \mathbf{\Pi}(\theta_2)$ *that satisfies* (LL) *only if* $\boldsymbol{\pi}^*(\theta_2) \in \mathbf{\Pi}(\theta_2)$.

The main implication of the corollary is that $\boldsymbol{\pi}^*(\theta_2) \neq \boldsymbol{\pi}^1(\theta_2)$, except in a very particular case. Not surprisingly, there is no change in the vectors of profits of the extreme types instead.

Let us again comment on the extreme types first. The fact that $\boldsymbol{\pi}^*(\theta_1) = \boldsymbol{\pi}^1(\theta_1)$ is explained similarly to the case with the binary signal. Let us see why $\boldsymbol{\pi}^*(\theta_3) = \boldsymbol{\pi}^1(\theta_3)$. Starting from $\pi_2(\theta_3) = \pi_3(\theta_3) = -L$, it is impossible to weaken the incentives of the lower types to report θ_3. Indeed, a decrease in $\pi_1(\theta_3)$ could only be induced together with a raise in either $\pi_2(\theta_3)$ or $\pi_3(\theta_3)$, which is not convenient for P, provided types θ_1 and θ_2 are both more likely to draw signal 2 and 3 than signal 1. Let us now consider the intermediate type. Interestingly, unlike for report θ_3, the incentive constraints whereby report θ_2 is unattractive to any other type are relaxed if $\pi_3(\theta_2)$ is set above $-L$ while, as we explained, $\pi_2(\theta_2) = -L$. To see this, replace $\pi_2(\theta_2) = -L$ in (20) and rearrange to obtain

$$\pi_3(\theta_2) \geq \frac{\Delta c(\theta_2, \theta_3) + L\frac{p_1(\theta_3) - p_1(\theta_2)}{p_1(\theta_2)}}{p_3(\theta_2)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}\right)} - L. \tag{24}$$

This shows that, if L is sufficiently high for (17) (or (16)) to be strictly satisfied, then $\pi_3(\theta_2)$ should be set strictly above $-L$. The reason is that this permits to account for the incentives of either extreme type to mimic the intermediate type. First, to discourage type θ_1 from over-reporting the best would be to set $\pi_2(\theta_2) = \pi_3(\theta_2) = -L$, since this type is less likely to draw signal 2 and 3 than signal 1. Yet, because $\pi_1(\theta_2)$ would be high in that case, type θ_3 might be motivated to under-report, provided this type is more likely to draw signal 1 than signal 2 and 3. This temptation is contrasted if one of the profits $\pi_2(\theta_2)$ and $\pi_3(\theta_2)$ is set above $-L$. The issue is then to establish which of those profits should be picked. If it is considered that the incentives of type θ_1 to over-report will be reinforced thereof, then the best is to pick the profit that has the weaker impact on those incentives. It turns out that, with $\psi_{2,3}(\theta_1, \theta_2) > \psi_{2,3}(\theta_3, \theta_2)$, that profit is $\pi_3(\theta_2)$. It remains to determine the values of $\pi_3(\theta_2)$ and $\pi_1(\theta_2)$. Replacing $\pi_2(\theta_2) = -L$ in (19) taken for $t = 2$ (hence, $t' = 1$) and rearranging, one obtains

$$\pi_3(\theta_2) \leq \frac{\Delta C(\theta_2, \theta_1) - L\frac{p_1(\theta_2) - p_1(\theta_1)}{p_1(\theta_2)}}{-p_3(\theta_2)\left(\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}\right)} - L. \tag{25}$$

Recall that (20), taken for $t = 2$ (and hence, $t' = 3$), is rewritten as (24) when $\pi_2(\theta_2) = -L$. Joint inspection of (24) and (25) evidences that, under (17) and (16), if there exists a range of feasible values of $\pi_3(\theta_2)$, then either it includes $-L$, or it lies entirely above $-L$. For instance, if P sets $\pi_3(\theta_2)$ in such a way as to saturate (25), and $\pi_1(\theta_2)$ is derived from (3) thereof, then the vector $\boldsymbol{\pi}^*(\theta_2)$ is obtained.

We now provide a numerical example to illustrate the results drawn in the setting with three informative signals.

Example 1 *Consider the Cobb-Douglas production function* $Q(l,k) = \frac{1}{\theta} l^{\alpha} k^{\beta}$. *Letting* P_l *and* P_k *be respectively the price of two inputs (say, labour and capital), the associated cost function is given by* $C(q,\theta) = g(\alpha,\beta) P_l^{\frac{\alpha}{\alpha+\beta}} P_k^{\frac{\beta}{\alpha+\beta}} (\theta q)^{\frac{1}{\alpha+\beta}}$, *where* $g(\alpha,\beta) \equiv \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}$ *and* q *is the level of production. Further consider the surplus function* $S(q) = \frac{A\varepsilon}{\varepsilon-1} q^{\frac{\varepsilon-1}{\varepsilon}}$, *where* $A > 0$, $\varepsilon > 0$. *In this setting,* $S'\left(q^{fb}(\theta)\right) = C_q\left(q^{fb}(\theta), \theta\right)$ *yields*

$$q^{fb}(\theta) = \left(\frac{1}{A} \frac{g(\alpha,\beta)}{\alpha+\beta} P_l^{\frac{\alpha}{\alpha+\beta}} P_k^{\frac{\beta}{\alpha+\beta}} \theta^{\frac{1}{\alpha+\beta}}\right)^{-\gamma},$$

where $\gamma \equiv \frac{\varepsilon(\alpha+\beta)}{\varepsilon+(1-\varepsilon)(\alpha+\beta)}$. *For an agent of type* θ' *the cost of producing* $q^{fb}(\theta)$ *units is*

$$C\left(q^{fb}(\theta), \theta'\right) = [A(\alpha+\beta)]^{\frac{\gamma}{\alpha+\beta}} \left(g(\alpha,\beta) P_l^{\frac{\alpha}{\alpha+\beta}} P_k^{\frac{\beta}{\alpha+\beta}}\right)^{1-\frac{\gamma}{\alpha+\beta}} (\theta')^{\frac{1}{\alpha+\beta}} \theta^{-\gamma}.$$

Take now the following values: $\alpha = 1.5$, $\beta = 1.3$, $\varepsilon = 0.8$, $P_l = 20$, $P_k = 30$, $a = 30$, $\theta_1 = 5$, $\theta_2 = 10$ *and* $\theta_3 = 14$; *and the following probability vectors, which satisfy property* (1) : $\mathbf{p}(\theta_1) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$, $\mathbf{p}(\theta_2) = (\frac{3}{4}, \frac{3}{16}, \frac{1}{16})$, $\mathbf{p}(\theta_3) = (\frac{5}{6}, \frac{3}{24}, \frac{1}{24})$. *Remark that, because* $\alpha+\beta = 2.8 > 1$, $C(q,\theta)$ *is concave in* θ. *With the previous values, we obtain* $\Delta C(\theta_2,\theta_1) \simeq 0.98$, $\Delta C(\theta_2,\theta_3) \simeq -0.56$, $p_1(\theta_2) - p_1(\theta_1) = \frac{1}{4}$, $p_1(\theta_3) - p_1(\theta_2) = \frac{1}{6}$. *Hence,* (6) *is violated and first best is not implemented with* $\boldsymbol{\pi}^1(\theta_2) = (2,-2,-2)$. *Instead, first best is implemented with* $\boldsymbol{\pi}^*(\theta_2)$ *as long as* $L \geq \Psi_{1,3}(\theta_1,\theta_2,\theta_3) = 47.2$. *Taking* $L = 47.2$, *we find* $\boldsymbol{\pi}^*(\theta_2) = (12.38, -47.2, 15.48)$.

4 A continuum of types and a finite number of signals

Take $\theta \in \Theta$ and $n \geq 3$. Given that the expected payoff of an agent of type θ', if he reports θ, is given by

$$\mathbb{E}_s\left[\widetilde{\pi}_s(\theta\,|\theta')\right] = \sum_{s=1}^{n} \pi_s(\theta)\, p_s(\theta') + \Delta C(\theta,\theta')$$

and that (3) must hold for first-best implementation, the incentive constraint is rewritten as

$$\Delta\theta(\theta,\theta') \leq \sum_{s=1}^{n} \pi_s(\theta)(p_s(\theta) - p_s(\theta')),\ \forall \theta',\theta \in \Theta. \tag{26}$$

Henceforth, we will refer to (26) as to the global incentive constraint in Γ. According to (26), θ is not an attractive report to type θ' if the gain in cost reimbursement associated with that lie, namely $\Delta C(\theta,\theta')$, is lower than the loss in terms of the lottery $(\sum_{s=1}^{n} \pi_s(\theta)(p_s(\theta) - p_s(\theta')))$, when $\theta > \theta'$; and if the loss in cost reimbursement, namely $-\Delta C(\theta,\theta')$, exceeds the gain in terms of the lottery $(\sum_{s=1}^{n} \pi_s(\theta)(p_s(\theta') - p_s(\theta)))$, when $\theta < \theta'$.

We hereafter determine the vectors of incentive compatible profits proceeding in a similar fashion as in the discrete case. Using (3), one can express $\pi_1(\theta)$ in terms of the profits assigned in all states other than 1. The expression so obtained can be used to reformulate (26) as the following pair of conditions on $\pi_n(\theta)$ (see Appendix B for details):[9]

$$\pi_n(\theta)\,p_n(\theta) \leq \frac{-\Delta C\left(\theta,\theta^-\right)}{\frac{p_n(\theta^-)}{p_n(\theta)}-\frac{p_1(\theta^-)}{p_1(\theta)}} - \sum_{s\neq 1,n}\pi_s(\theta)\,p_s(\theta)\psi_{s,n}\left(\theta^-,\theta\right),\ \forall\theta^- < \theta, \tag{27}$$

and

$$\pi_n(\theta)\,p_n(\theta) \geq \frac{\Delta C\left(\theta,\theta^+\right)}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}} - \sum_{s\neq 1,n}\pi_s(\theta)\,p_s(\theta)\psi_{s,n}\left(\theta^+,\theta\right),\ \forall\theta^+ > \theta. \tag{28}$$

Taking the limits for $\theta^- \to \theta$ and $\theta^+ \to \theta$ yields the local incentive constraint, formulated as the following expression of $\pi_n(\theta)$:[10]

$$\pi_n(\theta) = \frac{C_\theta(q^{fb}(\theta),\theta) + \sum_{s\neq 1,n}\pi_s(\theta)p_s(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_s'(\theta)}{p_s(\theta)}\right)}{-p_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_n'(\theta)}{p_n(\theta)}\right)},\ \forall\theta\in\Theta. \tag{29}$$

This tells that, once P chooses the profits to be assigned to type θ in states 2 to $n-1$, she must set the profit in state n according to (29) to prevent all neighboring types from reporting θ.

Once it is assessed that $\pi_1(\theta)$ must be chosen according to (3) and $\pi_n(\theta)$ such that (29) holds, it must be figured out how the profits should be set in states 2 to $n-1$ for (27) and (28) to be satisfied, taking into account that the limited liability constraint must hold in each state. Actually, the necessary conditions are of the kind already identified in the discrete-type case. That is, two potential conflicts must be eliminated. The first is that between the incentive constraints and the limited liability constraints. We will see that now, because types are drawn from a continuous interval, this boils down to a conflict between local incentive constraints and limited liability constraints. The second is the potential conflict between downward and upward incentive constraints.

First consider the former conflict. Looking at the incentive constraints, as formulated above, one can tell that this is a conflict between the upper bound imposed on $\pi_n(\theta)$ by (27) and the lower bound of $-L$ imposed by (LL). Besides, the choice of $\pi_n(\theta)$ depends also on how the profits are set in the remaining states. The local incentive constraint, as formulated in (29), defines the exact relationship between $\pi_n(\theta)$ and $\pi_s(\theta)$, for $s\neq 1,n$, such that the compensation scheme prevents any incentives to lie in a neighborhood. To account also for limited liability, one should first consider that, as is evident from (29), if any change is induced in $\pi_s(\theta)$, for some $s\neq 1,n$, then this change must be matched with an *opposite* variation in

[9] We let θ^- and θ^+ denote types respectively below and above θ, but not necessarily limit values around θ. We also adapt the notation previously introduced for the cost to the continuous case here considered, with analogous meaning.

[10] The standard procedure would be to derive the local incentive constraint directly from (26) by taking the limit for $\theta' \to \theta$. The expression of $\pi_n(\theta)$ in (29) would result after manipulating the local incentive constraint. We follow a somewhat different procedure that is functional to the analysis developed hereafter.

$\pi_n(\theta)$, and *vice versa.* Furthermore, an adjustment in $\pi_1(\theta)$ will be necessary to ensure that (3) holds. This is all summarized in the next lemma.

Lemma 1 *Take $n \geq 3$, $\boldsymbol{\pi}(\theta) \in \boldsymbol{\Pi}(\theta)$ $\forall\theta \in \Theta$, and any triplet of signals $\{i, j, k\} \in N$ such that*

$$\frac{p_i'(\theta)}{p_i(\theta)} > \frac{p_j'(\theta)}{p_j(\theta)} > \frac{p_k'(\theta)}{p_k(\theta)}, \ \forall\theta \in \Theta. \tag{30}$$

For any given value of $\pi_s(\theta) \in \boldsymbol{\pi}(\theta)$, $\forall s \notin \{i, j, k\}$, if a change is induced in $\pi_i(\theta)$, then changes must also be induced in $\pi_j(\theta)$ and $\pi_k(\theta)$, in two opposite directions, for the resulting vector $\boldsymbol{\pi}'(\theta)$ to be such that $\boldsymbol{\pi}'(\theta) \in \boldsymbol{\Pi}(\theta)$.

Being based on Lemma 1, one can conclude that the vector $\boldsymbol{\pi}(\theta)$, such that (LL) is most likely to be satisfied, is as follows. First, one sets $\pi_s(\theta) = \pi_n(\theta) < 0$, $\forall s \neq 1, n$, in such a way as to satisfy (29). Next, one sets $\pi_1(\theta) > 0$, in such a way as to extract all surplus. Not surprisingly, the resulting vector of profits is tantamount to $\boldsymbol{\pi}^1(\theta)$, the one derived by GBS.[11] The intuition behind this result is understood by interpreting Lemma 1 along the same lines as in the discrete-type analysis. Start from a situation in which $\pi_s(\theta) = \pi_n(\theta) < 0$, $\forall s \neq 1, n$, and local incentive compatibility is attained together with full surplus extraction. If $\pi_{s\neq 1}(\theta)$, say, is raised, then some other profit should be decreased to restore full surplus extraction. Suppose that profit is $\pi_1(\theta)$. Then, full surplus extraction can be restored, but the incentives to report θ of some type just below θ are strengthened, since that type is more likely to draw signal s than signal 1. The only way to also attain local incentive compatibility is to raise $\pi_1(\theta)$ and decrease some other profit, say $\pi_n(\theta)$. Hence, the limited liability constraint will be tightened. The following result obtains.

Lemma 2 *(GBS) $\exists\boldsymbol{\pi}(\theta) \in \boldsymbol{\Pi}(\theta)$ satisfying (LL) only if $\boldsymbol{\pi}^1(\theta)$ satisfies (LL).*

This result suggests that one vector of profits that could be used under limited liability is $\boldsymbol{\pi}^1(\theta)$. To establish whether this is or not the best choice, one should further account for the potential conflict between upward and downward incentive constraints. Recall that in the analysis of GBS this conflict is ruled out by assuming that the cost function is convex in the type, whereas the likelihood function of the reward signal is concave in the type. Indeed, under those conditions, first best is attained through the vector of profits $\boldsymbol{\pi}^1(\theta)$ $\forall\theta$, in line with Corollary 1.5 of RS.

As a first step, we use the expression of $\pi_n(\theta)$ in (29) to reformulate (27) and (28) as in the next lemma, where the following notation is used:

$$\Omega(\theta', \theta) \equiv \psi_{s,n}(\theta', \theta) - \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}, \ \forall\theta, \theta'.$$

[11] In their proof, GBS take any triplet of profits including the profit associated with signal 1 (*i.e.*, with the signal the conditional probability of which satisfies property (1)), and show that the limited liability constraints are most likely to be satisfied only if the other two profits are equal. Lemma 1 emphasizes that this is the case due to property (30), which will be useful for further analysis.

Lemma 3 *Given* (PC) *and* (29), (26) *is satisfied if and only if the following two conditions are satisfied for any given* $\theta \in (\underline{\theta}, \overline{\theta})$:

$$C_\theta\left(q^{fb}(\theta), \theta\right) \geq \left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)\left(\frac{\Delta C\left(\theta, \theta^-\right)}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} + \sum_{s\neq 1,n} \pi_s(\theta)\, p_s(\theta) \Omega\left(\theta^-, \theta\right)\right), \ \forall \theta^- < \theta, \tag{31}$$

and

$$C_\theta\left(q^{fb}(\theta), \theta\right) \leq \left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)\left(\frac{-\Delta C\left(\theta, \theta^+\right)}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} + \sum_{s\neq 1,n} \pi_s(\theta)\, p_s(\theta) \Omega\left(\theta^+, \theta\right)\right), \ \forall \theta^+ > \theta. \tag{32}$$

Being based on (31) and (32), and observing that $\Omega\left(\theta^-, \theta\right) - \Omega\left(\theta^+, \theta\right) = \psi_{s,n}\left(\theta^-, \theta\right) - \psi_{s,n}\left(\theta^+, \theta\right)$, we see that the necessary condition, to be verified for any triplet $\left\{\theta^-, \theta, \theta^+\right\}$, is written as

$$\frac{\Delta C\left(\theta, \theta^-\right)}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} - \frac{-\Delta C\left(\theta, \theta^+\right)}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} \leq -\sum_{s\neq 1,n} \pi_s(\theta)\, p_s(\theta) \left(\psi_{s,n}\left(\theta^-, \theta\right) - \psi_{s,n}\left(\theta^+, \theta\right)\right). \tag{33}$$

In good substance, one should first verify that, for each possible report θ, there exists a vector of profits such that (33) holds without violating (LL). Once this is ascertained, one should further verify that those profits satisfies (31) and (32) as well. Provided (31) has to be verified for any pair $\left\{\theta, \theta^+\right\}$, and (32) for any pair $\left\{\theta^-, \theta\right\}$, the analysis may look complex overall. The problem is tractable, in fact, thanks to the following result.

Lemma 4 *Condition* (33) *is necessary and sufficient for* (31) *and* (32) *to hold.*

The next step is to investigate how $\boldsymbol{\pi}(\theta)$ should be chosen such that (33) is weakest. To that end, it is useful to define

$$\rho_s(\theta', \theta) \equiv \frac{p_s(\theta') + (\theta - \theta')\, p_s'(\theta')}{p_s(\theta)}, \ \forall \theta' \neq \theta \in \Theta, \ \forall s \in N,$$

The magnitude of $\rho_s(\cdot, \cdot)$ is a measure of the curvature of the probability function of signal s. Indeed, $\rho_s(\theta', \theta) = 1$ if $p_s(\cdot)$ is linear, $\rho_s(\theta', \theta) < 1$ if $p_s(\cdot)$ is strictly convex, and $\rho_s(\theta', \theta) > 1$ if $p_s(\cdot)$ is strictly concave. Therefore, $\rho_s(\theta', \theta)$ can be used to assess how much the likelihood of signal s, if type is θ', diverges from the likelihood of that same signal, if type is θ. The more that $\rho_s(\theta', \theta)$ diverges from 1, the higher that the degree of convexity/concavity of $p_s(\cdot)$ is $\forall \theta' \neq \theta$, and the more that the likelihood of type θ' to draw signal s diverges from the likelihood of type θ to draw that same signal. It can be shown that if

$$\frac{\rho_s(\theta', \theta) - \rho_1(\theta', \theta)}{\left|\frac{p_1(\theta')}{p_1(\theta)} - \frac{p_s(\theta')}{p_s(\theta)}\right|} < \frac{\rho_n(\theta', \theta) - \rho_1(\theta', \theta)}{\left|\frac{p_1(\theta')}{p_1(\theta)} - \frac{p_n(\theta')}{p_n(\theta)}\right|}, \tag{34}$$

then the term in brackets in the right-hand side of (33) is negative $\forall\theta$ such that $\theta^- \leq \theta \leq \theta^+$, with at least one of these inequalities holding strictly (the proof is found in Appendix F).

Proposition 3 *Assume that* $n \geq 3$ *and* (34) *holds. The first-best allocation is implemented if and only if either*

$$\frac{\Delta C\left(\theta,\theta^-\right)}{-\Delta C\left(\theta,\theta^+\right)} \leq \frac{p_1(\theta)-p_1(\theta^-)}{p_1(\theta^+)-p_1(\theta)},\ \forall\theta,\theta^-,\theta^+\in\Theta,\ \theta^-<\theta<\theta^+ \tag{35}$$

and

$$L \geq \Delta C\left(\theta,\theta^-\right)\frac{p_1(\theta)}{p_1(\theta)-p_1(\theta^-)},\ \forall\theta^-,\theta\in\Theta,\ \theta^-<\theta, \tag{36}$$

or (35) *is violated and*

$$L \geq \Psi_{1,n}\left(\theta^-,\theta,\theta^+\right),\ \forall\theta,\theta^-,\theta^+\in\Theta,\ \theta^-<\theta<\theta^+. \tag{37}$$

This proposition extends Proposition 2 of GBS to a case where the cost is possibly concave in the type. In that case, the necessary condition (36) is no longer sufficient for first-best implementation.[12] If $\boldsymbol{\pi}^1(\theta)$ is used, then it is necessary to have also (35) satisfied. If (35) is found to be violated, then one should move away from $\boldsymbol{\pi}^1(\theta)$. This leads to a new necessary condition on the level of liability, namely (37). We denote as $\boldsymbol{\pi}^*(\theta)$ the vector of profits by means of which P is most likely to implement first best as long as (37) holds.

Corollary 3 *The vector of profits* $\boldsymbol{\pi}^*(\theta)$ *is composed as follows:*

$$\pi_1^*(\theta)=\frac{C_\theta\left(q^{fb}(\theta),\theta\right)-L\frac{p_n'(\theta)}{p_n(\theta)}}{p_1(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_n'(\theta)}{p_n(\theta)}\right)}-L,\ \pi_n^*(\theta)=\frac{L\frac{p_1'(\theta)}{p_1(\theta)}-C_\theta\left(q^{fb}(\theta),\theta\right)}{p_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_n'(\theta)}{p_n(\theta)}\right)}-L,\ \pi_{s\neq 1,n}^*(\theta)=-L,$$

where $\pi_1^*(\theta)>\pi_1^1(\theta),\ \pi_n^*(\theta)>\pi_n^1(\theta)$ *and* $\pi_s^*(\theta)<\pi_s^1(\theta),\ \forall s\neq 1,n,\ \forall\theta\in\left(\underline{\theta},\overline{\theta}\right).$

When the cost and the probability functions display the properties stated in Proposition 3, P should rely on Lemma 1 and proceed as follows. Starting from $\boldsymbol{\pi}^1(\theta)$, P should raise the profits in state 1 and n and decrease them in all other states. According to Lemma 1, P gains flexibility when switching from $\boldsymbol{\pi}^1(\theta)$ to a new vector, in which the profit in state 1 is raised and opposite changes are induced in the other profits. As explained in the discrete-type case, it is convenient to increase the profit of type θ in state 1 and decrease it in some state $s\neq 1$, because type θ^- is then led to face a greater loss, if it reports θ. This is because, under property (1), $\frac{p_1'(\theta)}{p_1(\theta)}>\frac{p_s'(\theta)}{p_s(\theta)},\ \forall s\neq 1$, involving that type θ^- will obtain less with a signal that it is more likely to draw, and more with a signal that it is less likely to draw. This process can be replicated for signal 1 and other $n-2$ signals, with which profits higher than $-L$ are

[12]Notice that as θ^- approaches θ (36) reduces to (9) for $i=1$, which is the exact formulation in GBS. We present the condition as in (36) because this alternative formulation helps us stress that the necessity of the condition only results from the incentives of lower types to exaggerate information.

initially associated. On the other hand, for one signal realization the profit must be increased in order to weaken the incentive of type θ^- to over-report. The remaining question is thus for which signal realization, beside 1, the profit should be increased, and for which ones it should be decreased instead. Corollary 3 identifies those signals.

Corollary 4 (31) *is relaxed and* (32) *is tightened when* $\boldsymbol{\pi}^*(\theta)$ *replaces* $\boldsymbol{\pi}^1(\theta)$.

This result formalizes the impossibility of lessening the incentives to over-report and the incentives to under-report altogether, by switching from one vector of profits to another in $\mathbf{\Pi}(\theta)$. However, provided that (34) holds, when replacing $\boldsymbol{\pi}^1(\theta)$ with $\boldsymbol{\pi}^*(\theta)$ the positive effect of type θ^- becoming less eager to claim θ prevails on the negative effect of type θ^+ becoming more eager to do that. Indeed, under (34), one has $\psi_{s,n}(\theta^-,\theta) > \psi_{s,n}(\theta^+,\theta)$, $\forall s \neq 1, n$, which is rewritten as

$$\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} < \frac{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_n(\theta^-)}{p_n(\theta)}}. \tag{38}$$

This is the counterpart of (22) in a setting with more than three types. Under (38), it is easier to lessen the conflict between the incentive constraints "from below" and "from above" if the profits of type θ are decreased to $-L$ in all states but 1 and n, rather than in all states but n only. Remarkably, when (33) is not a concern, as in the setting considered by GBS, it suffices to refer to the rate of change of the conditional probability to determine the profits that lessen the tension between local incentive compatibility and limited liability to the utmost. However, this is no longer the only requirement to be met in terms of probabilities, as it comes to the incentive scheme that makes the conflict between upward and downward incentive constraints weakest. The curvature of the function $p(\cdot)$ becomes important as well, because the potential gains and losses from the different lies depend on how the probabilities of the signals vary with the type. The next corollary lists conditions that are necessary and sufficient for (34) to hold and, hence, for (33) to be weakest.

Corollary 5 *For* (34) *to hold* $\forall s \neq 1, n$:

(*i*) *it is necessary to have* $\rho_s(\theta',\theta) < \max\{\rho_1(\theta',\theta), \rho_n(\theta',\theta)\}$ *and sufficient to have* $\rho_s(\theta',\theta) < \min\{\rho_1(\theta',\theta), \rho_n(\theta',\theta)\}$;

(*ii*) *it is necessary and sufficient to have*

$$\frac{\rho_n(\theta',\theta) - \rho_1(\theta',\theta)}{\rho_s(\theta',\theta) - \rho_1(\theta',\theta)} > \frac{\frac{p_1(\theta')}{p_1(\theta)} - \frac{p_n(\theta')}{p_n(\theta)}}{\frac{p_1(\theta')}{p_1(\theta)} - \frac{p_s(\theta')}{p_s(\theta)}}, \tag{39}$$

if

$$\rho_n(\theta',\theta) > \rho_s(\theta',\theta) > \rho_1(\theta',\theta);$$

and to have the converse of (39) *satisfied, if*

$$\rho_1(\theta',\theta) > \rho_s(\theta',\theta) > \rho_n(\theta',\theta).$$

Intuitively, because any decrease in $\pi_s(\theta)$, where $s \neq 1, n$, is compensated with an increase in both $\pi_1(\theta)$ and $\pi_n(\theta)$ (recall Lemma 1), $\boldsymbol{\pi}^*(\theta)$ cannot be employed unless at least one between $p_1(\cdot)$ and $p_n(\cdot)$ is less convex/more concave than the conditional probability of any other signal. If this is not the case, then the incentives to understate information are too strong for (33) to be lessened by means of $\boldsymbol{\pi}^*(\theta)$. Specifically, (32) is tightened more than (31) is relaxed (recall Corollary 4). The remaining conditions listed in Corollary 5, which are conditions on the degree of concavity/convexity of the likelihood functions, are sufficient for (34) to hold.

In substance, as long as (LL) does not bind in $\pi^1(\theta)$ at least for some θ, there is a neat benefit to P to move away from that vector of profits and exploit the slack of (LL).

Corollary 6 *Condition* (37) *is weaker than* (35).

In words, the benefit to P is that, under limited liability, incentive compatibility is attained for a wider family of cost functions than admissible under the sufficient condition of RS. This is immediately verified by reformulating (37) as

$$\frac{\Delta C\left(\theta,\theta^-\right)}{-\Delta C\left(\theta,\theta^+\right)} \leq \frac{p_1(\theta)-p_1\left(\theta^-\right)}{p_1\left(\theta^+\right)-p_1(\theta)} + \left(\frac{\frac{p_n(\theta^-)}{p_n(\theta)}-\frac{p_1(\theta^-)}{p_1(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}} - \frac{\frac{p_1(\theta)-p_1(\theta^-)}{p_1(\theta)}}{\frac{p_1(\theta^+)-p_1(\theta)}{p_1(\theta)}}\right) \tag{40}$$
$$+L\frac{\frac{p_n(\theta^-)}{p_n(\theta)}-\frac{p_1(\theta^-)}{p_1(\theta)}}{\Delta C\left(\theta,\theta^+\right)}\sum_{s\neq 1,n} p_s(\theta)\left(\psi_{s,n}\left(\theta^-,\theta\right)-\psi_{s,n}\left(\theta^+,\theta\right)\right),$$

and observing that the first term in the right-hand side is the same as in (35), and that the two additional terms are both positive. One should nonetheless recall that this benefit is available only if the liability is higher than required by GBS.

Corollary 7 *Condition* (37) *implies* (36) *if and only if* (35) *is violated.*

There is a simple conclusion to be drawn from this result. As long as (36) is slack, P can shift from $\boldsymbol{\pi}^1(\theta)$ to $\boldsymbol{\pi}^*(\theta)$ to take advantage of that slackness and relax the incentive compatibility constraints. Thereby, first best is at hand in a richer variety of contractual relationships.

5 A second-best analysis with discrete types

Being based on Proposition 3, we identify two possible departures from first best. The first occurs when (35) is satisfied whereas (36) is violated. In that case, which is considered by GBS in their second-best analysis, P cannot attain local incentive compatibility as long as she insists on the first-best allocation. The second departure from first best occurs when (35) is violated and, in addition, the maximum sustainable deficit (L) is not high enough to have (37) satisfied. We now turn to explore these two possibilities.[13]

To address these issues, we consider again a setting with three types and three signals. Our motivation for this focus is that, whereas standard solution methods are unlikely to be applicable in settings with a continuum of types, our first-best analysis made it plain that the incentives at work are neatly highlighted in the simple three-type framework.[14] Hence, we see no reason why the results we will derive with three types should not carry over with a continuum of types, once the technical complications are taken into account.

Formally, we take $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and $n = 3$. In a second-best setting, one expects at least one type to obtain an information rent, which would add complexity to the identification of the binding incentive constraints. Take the most efficient type, for instance. If this type must be given up a rent for not announcing θ_2, then one would want to design profits for type θ_2 in such a way as to minimize that rent. However, this cannot be done unless the lottery that type θ_3 faces, if it announces θ_2, is made more attractive; and it might also be necessary to concede a rent to dissuade type θ_3 from understating information. In turn, this may motivate type θ_1 to report θ_3 (rather than θ_2). Of course, this reasoning can be replicated, *mutatis mutandis*, for the least efficient type, to which report θ_1 might become more attractive than report θ_2.

To keep the analysis tractable and avoid complexities that would add little to the economic insights, we will restrict attention to situations where the extreme types are not tempted to mimic each other. To do so, we will solve a reduced programme of P, in which the associated incentive constraints are omitted. In a later stage, we will provide a sufficient condition for those constraints to be satisfied at the solution to the reduced problem, hence for that solution to be also a solution to the general problem Γ, in which all the incentive constraints are considered. The reduced problem is given by

$$\underset{\{q(\theta_t);\boldsymbol{\pi}(\theta_t),\forall\theta_t\}}{Max} \int_{\theta_t\in\{\theta_1,\theta_2,\theta_3\}} \left(S(q(\theta_t)) - C(q(\theta_t),\theta_t) - R(\theta_t)\right) f(\theta_t)$$

$$\text{subject to} \qquad (\Gamma')$$

$$R(\theta_t) \geq \Delta C(\theta_{t'},\theta_t) + \sum_s \pi_s(\theta_{t'}) p_s(\theta_t), \ \forall \{\theta_t,\theta_{t'}\} \neq \{\theta_1,\theta_3\}, \ \{\theta_t,\theta_{t'}\} \neq \{\theta_3,\theta_1\}$$

$$R(\theta_t) \geq 0, \ \forall\theta_t$$

$$\pi_s(\theta_t) \geq -L, \ \forall\theta_t, \ \forall s.$$

To characterize the solution to Γ', we will explore two alternative situations, in which either (36) or (37) is violated, taking into account that, with three types, (36) boils down to (16) and (17), and (37) to (23). If it is considered that (37) implies (36) (Corollary 7), then it is not surprising that the results presented below depend on whether (16) and (17) are violated, or

[13] Another possible departure from first best is that in which (34) is violated. However, in that case, the analysis would proceed along the same lines. That is, one would identify some pair of signals, other than $\{1, n\}$, the properties of which could be exploited in the design of the optimal profits.

[14] Technically speaking, the first-order approach may not be applicable because the contractual allocation is not necessarily differentiable.

these conditions are satisfied whereas (23) is violated. More importantly, the vectors of optimal profits differ in the two cases, similarly to the first-best framework.

Lemma 5 *At the solution to* Γ' :
(1) *If* (16) *and* (17) *are violated, then* $\pi_2(\theta_2) = \pi_3(\theta_2) = -L$.
(2) *If* (16) *and* (17) *are satisfied, whereas* (23) *is violated, then* $\pi_2(\theta_2) = -L < \pi_3(\theta_2)$.
In either case, $\pi_s(\theta_1) = -L$, $\forall s \neq 3$, *and* $\pi_s(\theta_3) = -L$, $\forall s \neq 1$.

Case (1) of the lemma, which arises when (36) is violated, is the counterpart of the result that GBS derive in their second-best analysis. Not surprisingly, type θ_2 is faced with the maximum loss in all states but one because this permits to minimize the rents that must be conceded to attain local incentive compatibility. Case (2) refers to situations where, rather than satisfying the local incentive constraints under limited liability, the relevant issue is to eliminate the conflict between upward and downward incentive constraints. Then, it is optimal for P to depart from a vector of profits with the characteristics previously described, just as switching from $\boldsymbol{\pi}^1(\theta_2)$ to $\boldsymbol{\pi}^*(\theta_2)$ was found to be optimal in the first-best analysis.

We now turn to investigate what information rents P concedes and what production levels she recommends, taking into account that the optimal profits are as presented in Lemma 5. We begin with case (1) in the lemma. Two effects are identified. First, the low types θ_1 and θ_2 cannot be prevented from over-reporting, unless they are conceded a rent. Even if they are exposed to the maximum sustainable deficit in states 2 and 3, which are both more likely to be drawn than state 1, namely $\pi_s(\theta_t) = -L$, $\forall \theta_t \neq \theta_1$, $\forall s \neq 1$, that deficit is not sufficiently high to discourage them from cheating, unless they are assigned a rent. Second, whereas setting $\pi_2(\theta_2) = \pi_3(\theta_2) = -L$ enables P to contain the rent of type θ_1 to the minimum affordable level, it might induce type θ_3 to report θ_2, unless it receives a rent. This issue makes the screening problem more complex, as we said. When it does not arise, the following result is obtained. With a slight abuse, we hereafter preserve the notation $\Delta C(\theta_t, \theta_{t'})$, $\forall \theta_t \neq \theta_{t'}$, to denote cost differences, with the understanding that it now refers to any production level $q(\theta_t)$ rather than to $q^{fb}(\theta_t)$ specifically.

Proposition 4 *Assume that* (16) *and* (17) *are violated and* $\psi_{2,3}(\theta_1, \theta_2) > \psi_{2,3}(\theta_3, \theta_2)$. *Further assume that* $R(\theta_3) = 0$ *at the solution to* Γ'. *Then, the optimal information rents are given by*

$$R(\theta_1) = \Delta C(\theta_2, \theta_1) + \Delta C(\theta_3, \theta_2) \frac{p_1(\theta_1)}{p_1(\theta_2)} - L \frac{p_1(\theta_3) - p_1(\theta_1)}{p_1(\theta_3)} \tag{41}$$

$$R(\theta_2) = \Delta C(\theta_3, \theta_2) - L \frac{p_1(\theta_3) - p_1(\theta_2)}{p_1(\theta_3)}. \tag{42}$$

The optimal production levels are such that $q(\theta_1) = q^{fb}(\theta_1)$, $q(\theta_2) < q^{fb}(\theta_2)$, $q(\theta_3) < q^{fb}(\theta_3)$ *and*

$$\frac{-\Delta C(\theta_2, \theta_3)}{\Delta C(\theta_3, \theta_2)} \geq \frac{p_1(\theta_3)}{p_1(\theta_2)}. \tag{43}$$

The expressions in (41) and (42) are tantamount to those obtained by GBS in their second-best analysis. Moreover, as in their solution, the quantities of all types but the most efficient one are both decreased below their first-best levels to contain the rents, which is standard in screening problems. In addition, those quantities must be such that (43) is satisfied. Absent any restriction on the shape of the cost function with respect to the type, (43) is found to be necessary for types θ_2 and θ_3 not being tempted to mimic each other. This condition requires setting $q(\theta_2)$ sufficiently high relative to $q(\theta_3)$. With this, the penalty of $-\Delta C(\theta_2, \theta_3)$, which type θ_3 faces if it under-reports, is sufficiently high relative to the bonus of $\Delta C(\theta_3, \theta_2)$, which type θ_2 faces if it over-reports, to eliminate any incentives to deliver those fake reports.

We now provide a sufficient condition for the second effect aforementioned to be ruled out so that $R(\theta_3) = 0$ at the solution to Γ'. Actually, it is a condition usually assumed in the literature, and our results confirm previous findings, in fact.

Remark 1 *The solution with* $R(\theta_3) = 0$ *presented in Proposition* 4, *arises if*

$$\frac{\Delta C(\theta_2, \theta_1)}{-\Delta C(\theta_2, \theta_3)} < \frac{p_1(\theta_2) - p_1(\theta_1)}{p_1(\theta_3) - p_1(\theta_2)}. \tag{44}$$

Intuitively, if type θ_3 announces θ_2, rather than telling the truth, it faces an increase in the expected value of the lottery, due to the fact that it receives a reward in the state it is most likely to draw - state 1. If the cost is either more convex or less concave with respect to the type, relative to the likelihood of signal 1, then the increase in the value of the lottery faced by type θ_3 is more than offset by the penalty in terms of cost, which is high under (43), as we said. It follows that (6) holds true, which explains why type θ_3 is discouraged from under-reporting without receiving any rent.

Let us next consider case (2) in Lemma 5, where the level of liability is high enough to satisfy (17) and (16) but not (23). Similarly to case (1), there are two effects potentially at work. First, P cannot make sure that types θ_1 and θ_3 are unwilling to mimic each other, unless she concedes an information rent at least to one such type. To contain that rent, she distorts the production of type θ_2 away from its first-best level. Second, if type θ_1 receives a rent, then type θ_3 might find report θ_1 more attractive than report θ_2; similarly, if type θ_3 receives a rent, then type θ_1 might find report θ_3 more attractive than report θ_2. However, given our focus on Γ', the results presented hereafter are those obtained if the second effect is not at work.

Before characterizing this contractual solution, it is useful to denote $\Delta\psi \equiv \psi_{2,3}(\theta_1, \theta_2) - \psi_{2,3}(\theta_3, \theta_2)$ and $\chi \equiv \frac{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}$.

Proposition 5 *Assume that* (17) *and* (16) *are satisfied, whereas* (23) *is violated. Then, at the solution to* Γ', $R(\theta_2) = 0$. *Furthermore:*

(1) *If* $\frac{f(\theta_3)}{f(\theta_1)} > \chi$, *then* $R(\theta_3) = 0$ *and* $R(\theta_1) = R_1^{sb}$, *where:*

$$R_1^{sb} \equiv \left(\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}\right) p_2(\theta_2)\, \Delta\psi\left(\Psi_{1,3}(\theta_1, \theta_2, \theta_3) - L\right).$$

(2) *If* $\frac{f(\theta_3)}{f(\theta_1)} < \chi$*, then* $R(\theta_1) = 0$ *and* $R(\theta_3) = R_3^{sb}$*, where:*

$$R_3^{sb} \equiv \left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}\right) p_2(\theta_2)\,\Delta\psi\left(\Psi_{1,3}(\theta_1,\theta_2,\theta_3) - L\right).$$

(3) *If* $\frac{f(\theta_3)}{f(\theta_1)} = \chi$*, then* $R(\theta_1) \in \left(0, R_1^{sb}\right)$ *and* $R(\theta_3) \in \left(0, R_3^{sb}\right)$ *such that* $R(\theta_1) = R_1^{sb} - \chi R(\theta_3)$*.*

The optimal production levels are such that $q(\theta_t) = q^{fb}(\theta_t)$*,* $\forall t = 1, 3$*, and* $q(\theta_2) < q^{fb}(\theta_2)$ *if and only if*

$$\frac{d}{dq}\left(\frac{\Delta C(\theta_2,\theta_1)}{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} - \frac{-\Delta C(\theta_2,\theta_3)}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}\right) > 0.$$

Interestingly, considering the three cases in the proposition, we see that, except in a particular case, P concedes an information rent to only one type. To interpret, consider the profits assigned to type θ_2 and the incentives they provide to the extreme types, and reason along the lines of the first-best analysis. First, because $\Delta\psi > 0$ (given property (22)), it is optimal to set $\pi_2(\theta_2) = -L$ to make the conflict between upward and downward incentive constraints weakest. Next, it is necessary to establish how $\pi_1(\theta_2)$ and $\pi_3(\theta_2)$ should be set, considering that, given $R(\theta_2) = 0$, the higher that one such profit is the lower that the other must be. An increase in $\pi_3(\theta_2)$, associated with a decrease in $\pi_1(\theta_2)$, triggers an increase in $R(\theta_1)$ and a decrease in $R(\theta_3)$. Intuitively, because type θ_1 is more likely to draw signal 3 and type θ_3 is more likely to draw signal 1, these changes make report θ_2 more attractive to the former type, which requires conceding a higher rent to prevent cheating, and less attractive to the latter type, which allows for a lower rent. Obviously, the converse is true if a decrease is induced in $\pi_3(\theta_2)$, as coupled with an increase in $\pi_1(\theta_2)$. In substance, $\pi_1(\theta_2)$ and $\pi_3(\theta_2)$ are used as tools to reduce $R(\theta_1)$, at the cost of inducing an increase in $R(\theta_3)$, or *vice versa*. To preserve incentive compatibility, a variation of one unit in $R(\theta_3)$ must be matched with a variation of χ units in $R(\theta_1)$ in the opposite direction. Because (23) is violated, entailing that there are no values of $\pi_3(\theta_2)$ such that (24) is satisfied together with (25), that profit must be picked as follows:

$$\pi_3(\theta_2) \in \left[\frac{-\Delta C(\theta_2,\theta_1) + L\frac{p_1(\theta_2)-p_1(\theta_1)}{p_1(\theta_2)}}{p_3(\theta_2)\left(\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}\right)} - L,\ \frac{\Delta C(\theta_2,\theta_3) + L\frac{p_1(\theta_3)-p_1(\theta_2)}{p_1(\theta_2)}}{p_3(\theta_2)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}\right)} - L\right].$$

In case (1), where the frequency of type θ_3 is high relative to that of type θ_1, and the rate of adjustment of $R(\theta_1)$ to changes in $R(\theta_3)$ is low, it is least costly for P if she sets $\pi_3(\theta_2)$ to the highest value of the feasible range and avoids conceding a rent to type θ_3. In case (2), where the opposite conditions are true, P prefers to set $\pi_3(\theta_2)$ to the lowest value of the feasible range so that $R(\theta_1) = 0$ instead. In case (3), any value of $\pi_3(\theta_2)$ in the feasible range is indifferent to P because any associated values of the rents $R(\theta_1)$ and $R(\theta_3)$, determined as previously explained, are equally costly to her.

To complete the analysis, it must be verified that the incentive constraints whereby the extreme types are unwilling to mimic each other, which were omitted in Γ', are satisfied. It turns out that a sufficient condition for this is the cost function not being "too concave" with respect to the type.

Corollary 8 $\exists \phi > 0$ *such that if*

$$\frac{\Delta C(\theta_2, \theta_1)}{-\Delta C(\theta_2, \theta_3)} \leq \phi, \tag{45}$$

then: (i) *at the solution to* Γ', *the assumption* $R(\theta_3) = 0$ *of Proposition* 4 *is satisfied;* (ii) *the solution to* Γ' *is a solution to* Γ.

As long as (45) holds, the gain of $\Delta C(\theta_2, \theta_1)$ that type θ_1 obtains, if it announces θ_2, is sufficiently low relative to the penalty of $-\Delta C(\theta_2, \theta_3)$ that type θ_3 faces, if it announces θ_2. Hence, preventing any of the extreme types from announcing the intermediate type is relatively cheap, *i.e.*, it does not require giving up much surplus. As a result, there is no incentive for any of the extreme types to mimic the other, if the other obtains a rent under the conditions of Proposition 4 and 5. For analogous reason, there is no need to assign a rent to the least efficient type in case (1) of Lemma 5, as we explained.

6 Conclusion

In a principal-agent model with private information on cost and *ex-post* public signals, we showed that the focus on the full-rank condition, the most common approach in the literature, is not necessarily the best approach to study optimal contractual design. This is due to the agent being protected by limited liability. Because of that, one cannot abstract from considering the characteristics of the cost function to assess what contractual attainments are at reach. Whereas Bose and Zhao [1] identify conditions for first-best implementation when the full-rank condition does not hold, our contribution is to show that, under limited liability, the possibility of attaining the first-best outcome is not necessarily determined by the way in which the conditional probabilities of the signals depart from the full-rank condition. Moreover, the existence of an exact relationship between the extent of the liability and the admissible degree of concavity of the cost function (when this is not convex in the type) involves that the set of technologies for which first best is at reach under limited liability is richer than considered by GBS.

We highlighted that the principal is faced with two essential issues. First, low (efficient) types should be faced with sufficiently unfair lotteries, as meant to discourage them from exaggerating cost. Second, high (inefficient) types should not be faced with particularly attractive lotteries, which might incentivize them to pretend a lower cost. We found that the contractual solution - whether the first-best or a second-best one - depends on which issue is more concerning for the principal. When the second issue is more important, a given contractual performance is attained with a higher level of liability than would be otherwise. Accordingly, the level of liability that separates the situations in which the first-best outcome is viable from those in which it is not, also separates the situations in which the first issue is preeminent from those in which the second is preeminent instead. Thus, relative to the existing literature, the results of our analysis offer a wider perspective on the optimal contractual design with informative signals.

As a general view, our study contributes to shedding light on how to attain incentive compatibility when the principal faces more than two types of agent and can use more than two signals in contracting. Our findings suggest that, when exploring agency relationships with informative signals and limited liability, it might be with loss of generality to restrict attention to the two-type case, or to a binary signal. Nonetheless, by allowing for more than two signals, we found that it is not only necessary but also sufficient to select only three of them, which display desirable properties, regardless of the exact available number. This points to the conclusion that a parsimonious but appropriate use of exogenous information is enough to enhance contracting in a variety of practical instances.

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A The vector $\mathbf{h}(\theta_t)$ under property (1)

We show that property (1) implies (10) and (11) for θ_1 and θ_T, and that it does not imply (10) and (11) for the other types.

Suppose $\nexists \mathbf{h}(\theta_1)$ as defined in the text. It follows from Farkas' lemma that $\mathbf{p}(\theta_1)$ is not linearly independent of the other vectors of probabilities. Then, there exists a vector $(\lambda_2, ..., \lambda_T)$, where $\lambda_t \in [0,1]$, $\forall t \in \{2, ..., T\}$, and $\sum_{t=2}^{T} \lambda_t = 1$, such that

$$p_s(\theta_1) = \lambda_2 p_s(\theta_2) + ... + \lambda_T p_s(\theta_T), \ \forall s \in N.$$

Using this condition for $s = 1$ and $s \neq 1$, jointly with property (1), we obtain

$$p_1(\theta_1) = \lambda_2 p_1(\theta_2) + ... + \lambda_T p_1(\theta_T)$$

or, equivalently,

$$\begin{aligned} \frac{p_1(\theta_1)}{p_1(\theta_2)} &= \lambda_2 \frac{p_1(\theta_2)}{p_1(\theta_2)} + ... + \lambda_T \frac{p_1(\theta_T)}{p_1(\theta_2)} \\ &> \lambda_2 \frac{p_s(\theta_2)}{p_s(\theta_2)} + ... + \lambda_T \frac{p_s(\theta_T)}{p_s(\theta_2)} = \frac{p_s(\theta_1)}{p_s(\theta_2)}. \end{aligned}$$

The inequality $\frac{p_1(\theta_1)}{p_1(\theta_2)} > \frac{p_s(\theta_1)}{p_s(\theta_2)}$ contradicts property (1) and, hence, the hypothesis that $\nexists \mathbf{h}(\theta_1)$. Therefore, (1) implies $\mathbf{h}(\theta_1)$.

Similarly, suppose $\nexists \mathbf{h}(\theta_T)$. Then, there exists a vector $(\lambda_1, ..., \lambda_{T-1})$, where $\lambda_t \in [0,1]$, $\forall t \in \{1, ..., T-1\}$, and $\sum_{t=1}^{T-1} \lambda_t = 1$, such that

$$p_s(\theta_T) = \lambda_1 p_s(\theta_1) + ... + \lambda_{T-1} p_s(\theta_{T-1}), \ \forall s \in N.$$

Using this for $s = 1$ and $s \neq 1$, jointly with property (1), we obtain

$$p_1(\theta_T) = \lambda_1 p_1(\theta_1) + ... + \lambda_{T-1} p_1(\theta_{T-1})$$

or, equivalently,

$$\begin{aligned} \frac{p_1(\theta_T)}{p_1(\theta_{T-1})} &= \lambda_1 \frac{p_1(\theta_1)}{p_1(\theta_{T-1})} + ... + \lambda_{T-1} \frac{p_1(\theta_{T-1})}{p_1(\theta_{T-1})} \\ &< \lambda_1 \frac{p_s(\theta_1)}{p_s(\theta_{T-1})} + ... + \lambda_{T-1} \frac{p_s(\theta_{T-1})}{p_s(\theta_{T-1})} = \frac{p_s(\theta_T)}{p_s(\theta_{T-1})}. \end{aligned}$$

The inequality $\frac{p_1(\theta_T)}{p_1(\theta_{T-1})} < \frac{p_s(\theta_T)}{p_s(\theta_{T-1})}$ contradicts property (1) and, hence, also the initial hypothesis that $\nexists \mathbf{h}(\theta_T)$. Therefore, $\mathbf{h}(\theta_T)$ is implied by property (1).

Take any vector $\mathbf{p}(\theta_t)$, where $t \notin \{1, T\}$. Suppose $\exists \mathbf{h}(\theta_t)$. Then, there is no vector $(\lambda_1, ..., \lambda_{t-1}, \lambda_{t+1}, ..., \lambda_T)$, where $\lambda_t \in [0,1]$ $\forall t \in \{1, ..., t-1, t+1, ..., T\}$ and $\sum_{t' \neq t} \lambda_{t'} = 1$, such that

$$p_s(\theta_t) = \lambda_1 p_s(\theta_1) + ... + \lambda_{t-1} p_s(\theta_{t-1}) + \lambda_{t+1} p_s(\theta_{t+1}) + ... + \lambda_T p_s(\theta_T),$$

which is equivalent to

$$\frac{p_s(\theta_t)}{p_s(\theta_{t+1})} = \lambda_1 \frac{p_s(\theta_1)}{p_s(\theta_{t+1})} + ... + \lambda_{t-1} \frac{p_s(\theta_{t-1})}{p_s(\theta_{t+1})} + \lambda_{t+1} \frac{p_s(\theta_{t+1})}{p_s(\theta_{t+1})} + ... + \lambda_T \frac{p_s(\theta_T)}{p_s(\theta_{t+1})}. \tag{46}$$

However, taking $\mathbf{p}(\theta_t)$ such that

$$\begin{aligned} & \frac{p_{s'}(\theta_t)}{p_{s'}(\theta_{t+1})} \\ > & \frac{p_1(\theta_t)}{p_1(\theta_{t+1})} \\ > & \frac{p_{s'}(\theta_t)}{p_{s'}(\theta_{t+1})} + \lambda_1 \left(\frac{p_1(\theta_1)}{p_1(\theta_{t+1})} - \frac{p_{s'}(\theta_1)}{p_{s'}(\theta_{t+1})} \right) + ... + \lambda_{t-1} \left(\frac{p_1(\theta_{t-1})}{p_1(\theta_{t+1})} - \frac{p_{s'}(\theta_{t-1})}{p_{s'}(\theta_{t+1})} \right), \ \forall s' \neq 1, \end{aligned}$$

it is found that both property (1) and the equality in (46) are satisfied. To see this, first use (46) for $s = 1$ to rewrite the second inequality here above as

$$\begin{aligned} \lambda_{t+1} \frac{p_1(\theta_{t+1})}{p_1(\theta_{t+1})} + ... + \lambda_T \frac{p_1(\theta_T)}{p_1(\theta_{t+1})} & > \frac{p_{s'}(\theta_t)}{p_{s'}(\theta_{t+1})} + \lambda_1 \left(-\frac{p_{s'}(\theta_1)}{p_{s'}(\theta_{t+1})} \right) + ... + \lambda_{t-1} \left(-\frac{p_{s'}(\theta_{t-1})}{p_{s'}(\theta_{t+1})} \right) \\ & = \lambda_{t+1} \frac{p_{s'}(\theta_{t+1})}{p_{s'}(\theta_{t+1})} + ... + \lambda_T \frac{p_{s'}(\theta_T)}{p_{s'}(\theta_{t+1})}. \end{aligned}$$

Next use (46) for s' to rewrite

$$\lambda_{t+1} \frac{p_1(\theta_{t+1})}{p_1(\theta_{t+1})} + ... + \lambda_T \frac{p_1(\theta_T)}{p_1(\theta_{t+1})} > \lambda_{t+1} \frac{p_{s'}(\theta_{t+1})}{p_{s'}(\theta_{t+1})} + ... + \lambda_T \frac{p_{s'}(\theta_T)}{p_{s'}(\theta_{t+1})},$$

which is true by property (1). Hence, there exists a vector $(\lambda_1, ..., \lambda_{t-1}, \lambda_{t+1}, ..., \lambda_T)$ and the initial hypothesis is contradicted, entailing that $\nexists \mathbf{h}(\theta_t), \forall t \neq 1, T$.

B Derivation of (27) and (28)

Using $\widetilde{\pi}_s(\theta \,|\theta') = z_s(\theta) - C(q(\theta), \theta')$ and $\pi_s(\theta) = \widetilde{\pi}_s(\theta \,|\theta)$, we have

$$\mathbb{E}_s[\widetilde{\pi}_s(\theta \,|\theta')] = \sum_{s=1}^{n} \pi_s(\theta) p_s(\theta') + C(q(\theta), \theta) - C(q(\theta), \theta').$$

Using (3), $\mathbb{E}_s[\widetilde{\pi}_s(\theta \,|\theta')]$ is rewritten as (26). Rewriting (3) as $\pi_1(\theta) = -\sum_{s=2}^{n} \pi_s(\theta) \frac{p_s(\theta)}{p_1(\theta)}$, (26) is rewritten as

$$\Delta C(\theta, \theta') \leq \sum_{s \neq 1, n} \pi_s(\theta) p_s(\theta) \left(\frac{p_1(\theta')}{p_1(\theta)} - \frac{p_s(\theta')}{p_s(\theta)} \right) + \pi_n(\theta) p_n(\theta) \left(\frac{p_1(\theta')}{p_1(\theta)} - \frac{p_n(\theta')}{p_n(\theta)} \right),$$

hence as

$$\pi_n(\theta) p_n(\theta) \left(\frac{p_1(\theta')}{p_1(\theta)} - \frac{p_n(\theta')}{p_n(\theta)} \right) \geq \Delta C(\theta, \theta') - \sum_{s \neq 1,n} \pi_s(\theta) p_s(\theta) \left(\frac{p_1(\theta')}{p_1(\theta)} - \frac{p_s(\theta')}{p_s(\theta)} \right). \quad (47)$$

Recall that, by assumption, $\frac{p_1(\theta')}{p_1(\theta)} > \frac{p_n(\theta')}{p_n(\theta)}$ if and only if $\theta' > \theta$. Using this property for $\theta^- < \theta$ and $\theta^+ > \theta$, (47) is respectively rewritten as (27) and (28).

C Proof of Lemma 1

Recall $\widetilde{\pi}_s(\theta | \theta') = z_s(\theta) - C\left(q^{fb}(\theta), \theta'\right)$ and

$$\mathbb{E}_s[\widetilde{\pi}_s(\theta | \theta')] \equiv \sum_{s=1}^{n} \left(z_s(\theta) - C\left(q^{fb}(\theta), \theta'\right)\right) p_s(\theta'). \quad (48)$$

The first-order condition of the agent's problem, evaluated at $\theta' = \theta$, is given by:

$$\sum_{s=1}^{n} \left(z_s'(\theta) - C_q\left(q^{fb}(\theta), \theta\right) \left(q^{fb}(\theta)\right)' \right) p_s(\theta) = 0. \quad (49)$$

From $z_s(\theta) = \pi_s(\theta) + C\left(q^{fb}(\theta), \theta\right)$, we compute $z_s'(\theta) = \pi_s'(\theta) + C_q\left(q^{fb}(\theta), \theta\right)\left(q^{fb}(\theta)\right)' + C_\theta\left(q^{fb}(\theta), \theta\right)$, which we then replace into (49) to find

$$C_\theta\left(q^{fb}(\theta), \theta\right) = -\sum_{s=1}^{n} \pi_s'(\theta) p_s(\theta). \quad (50)$$

Using (3), we obtain $-\sum_{s=1}^{n} \pi_s'(\theta) p_s(\theta) = \sum_{s=1}^{n} \pi_s(\theta) p_s'(\theta)$. Using this equality, (50) is rewritten as

$$C_\theta\left(q^{fb}(\theta), \theta\right) = \sum_{s=1}^{n} \pi_s(\theta) p_s'(\theta) \quad (51)$$

Suppose that some profit $\pi_i(\theta)$ is changed by ε_i. Accordingly, $\pi_j(\theta)$ is changed by ε_j and $\pi_k(\theta)$ by ε_k such that (PC) is still saturated and the right-hand side of (51) does not vary. Dropping the argument θ everywhere for the sake of shortness, this requires

$$\varepsilon_j p_j = -\varepsilon_i p_i - \varepsilon_k p_k \Leftrightarrow \varepsilon_j = -\varepsilon_i \frac{p_i}{p_j} - \varepsilon_k \frac{p_k}{p_j} \quad (52)$$

$$\varepsilon_k p_k' = -\varepsilon_j p_j' - \varepsilon_i p_i' \Leftrightarrow \varepsilon_k = -\varepsilon_j \frac{p_j'}{p_k'} - \varepsilon_i \frac{p_i'}{p_k'}. \quad (53)$$

Replacing first ε_k from (53) in (52), then ε_j from (52) in (53), and rearranging we obtain

$$\varepsilon_j = -\varepsilon_i \frac{p_i}{p_j} \frac{\frac{p_i'}{p_i} - \frac{p_k'}{p_k}}{\frac{p_j'}{p_j} - \frac{p_k'}{p_k}}, \quad \varepsilon_k = \varepsilon_i \frac{p_i}{p_k} \frac{\frac{p_i'}{p_i} - \frac{p_j'}{p_j}}{\frac{p_j'}{p_j} - \frac{p_k'}{p_k}}, \quad (54)$$

from which we deduce that ε_j and ε_k have different signs.

D Proof of Lemma 3

Taking the expression of $\pi_n(\theta)\, p_n(\theta)$ from (29), plugging into (27) and making use of the inequalities $\frac{p_1'(\theta)}{p_1(\theta)} > \frac{p_n'(\theta)}{p_n(\theta)}$ and $\frac{p_n(\theta^-)}{p_n(\theta)} > \frac{p_1(\theta^-)}{p_1(\theta)}$ to rearrange, (27) is rewritten as (31). Similarly, (28) is rewritten as (32).

E Proof of Lemma 4

The necessity of (33) is obvious. To show sufficiency, we compute

$$\lim_{\theta' \to \theta} \psi_{s,n}(\theta', \theta) = \lim_{\theta' \to \theta} \frac{\frac{\frac{p_1(\theta') - p_1(\theta)}{\theta' - \theta}}{p_1(\theta)} - \frac{\frac{p_s(\theta') - p_s(\theta)}{\theta' - \theta}}{p_s(\theta)}}{\frac{\frac{p_1(\theta') - p_1(\theta)}{\theta' - \theta}}{p_1(\theta)} - \frac{\frac{p_n(\theta') - p_n(\theta)}{\theta' - \theta}}{p_n(\theta)}} = \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}. \tag{55}$$

Taking $\theta^+ \to \theta$ and using (55), (33) is rewritten as (31). Similarly, taking $\theta^- \to \theta$ and using (55), (33) is rewritten as (32). Hence, (33) boils down to either (31) or (32), entailing that it is sufficient for either such condition to hold.

F Proof of Proposition 3

Derivation of (37)

Using the definition of $\psi_{s,n}(\theta^+, \theta)$, we see that $\frac{d\psi_{s,n}(\theta^+,\theta)}{d\theta^+} < 0$ if and only if

$$\frac{\frac{p_1'(\theta^+)}{p_1(\theta)} - \frac{p_s'(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}} < \frac{\frac{p_1'(\theta^+)}{p_1(\theta)} - \frac{p_n'(\theta^+)}{p_n(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}. \tag{56}$$

Multiplying the numerator by $(\theta^+ - \theta)$ in both sides, subtracting 1 from each side and manipulating further, (56) becomes

$$\frac{\frac{p_s(\theta^+) - p_s'(\theta^+)(\theta^+ - \theta)}{p_s(\theta)} - \frac{p_1(\theta^+) - p_1'(\theta^+)(\theta^+ - \theta)}{p_1(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}} < \frac{\frac{p_n(\theta^+) - p_n'(\theta^+)(\theta^+ - \theta)}{p_n(\theta)} - \frac{p_1(\theta^+) - p_1'(\theta^+)(\theta^+ - \theta)}{p_1(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}.$$

Using the definition of $\rho_s(\theta', \theta)$, this is rewritten as

$$\frac{\rho_s(\theta^+, \theta) - \rho_1(\theta^+, \theta)}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}} < \frac{\rho_n(\theta^+, \theta) - \rho_1(\theta^+, \theta)}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}, \tag{57}$$

which is satisfied by assumption. Similarly, using the definition of $\psi_{s,n}\left(\theta^{-},\theta\right)$, we see that $\frac{d\psi_{s,n}\left(\theta^{-},\theta\right)}{d\theta^{-}}<0$ if and only if

$$\frac{\frac{p_1'\left(\theta^{-}\right)}{p_1(\theta)}-\frac{p_s'\left(\theta^{-}\right)}{p_s(\theta)}}{\frac{p_s\left(\theta^{-}\right)}{p_s(\theta)}-\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}}>\frac{\frac{p_1'\left(\theta^{-}\right)}{p_1(\theta)}-\frac{p_n'\left(\theta^{-}\right)}{p_n(\theta)}}{\frac{p_n\left(\theta^{-}\right)}{p_n(\theta)}-\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}}. \tag{58}$$

Multiplying the numerator by $\left(\theta-\theta^{-}\right)$ in both sides, subtracting 1 from either side and rearranging, (58) becomes

$$\frac{\frac{p_1\left(\theta^{-}\right)+p_1'\left(\theta^{-}\right)\left(\theta-\theta^{-}\right)}{p_1(\theta)}-\frac{p_s\left(\theta^{-}\right)+p_s'\left(\theta^{-}\right)\left(\theta-\theta^{-}\right)}{p_s(\theta)}}{\frac{p_s\left(\theta^{-}\right)}{p_s(\theta)}-\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}}>\frac{\frac{p_1\left(\theta^{-}\right)+p_1'\left(\theta^{-}\right)\left(\theta-\theta^{-}\right)}{p_1(\theta)}-\frac{p_n\left(\theta^{-}\right)+p_n'\left(\theta^{-}\right)\left(\theta-\theta^{-}\right)}{p_n(\theta)}}{\frac{p_n\left(\theta^{-}\right)}{p_n(\theta)}-\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}}.$$

Resting on the definition of ρ, this is rewritten as

$$\frac{\rho_s\left(\theta^{-},\theta\right)-\rho_1\left(\theta^{-},\theta\right)}{\frac{p_s\left(\theta^{-}\right)}{p_s(\theta)}-\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}}<\frac{\rho_n\left(\theta^{-},\theta\right)-\rho_1\left(\theta^{-},\theta\right)}{\frac{p_n\left(\theta^{-}\right)}{p_n(\theta)}-\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}}, \tag{59}$$

which is satisfied by assumption. Therefore, we have $\frac{d\psi_{s,n}\left(\theta^{+},\theta\right)}{d\theta^{+}}<0$ and $\frac{d\psi_{s,n}\left(\theta^{-},\theta\right)}{d\theta^{-}}<0$, so that the difference $\psi_{s,n}\left(\theta^{-},\theta\right)-\psi_{s,n}\left(\theta^{+},\theta\right)$ has a lower bound for $\theta^{-}\to\theta$ and $\theta^{+}\to\theta$. Using (55), the lower bound is found to be zero. Hence, $\psi_{s,n}\left(\theta^{-},\theta\right)-\psi_{s,n}\left(\theta^{+},\theta\right)\geq 0$, $\forall\left\{\theta^{-},\theta,\theta^{+}\right\}$ such that $\theta^{-}<\theta<\theta^{+}$, and (33) is weakest if $\pi_s(\theta)=-L$, $\forall s\neq 1,n$. Replacing in (33) and rearranging yields (37).

Derivation of (35) and (36)

Setting $\pi_s(\theta)=\pi_n(\theta)$ in (27), we see that $\pi_n(\theta)\geq -L$ if and only if (36) is satisfied. It follows immediately from Lemma 1 that, if $\boldsymbol{\pi}^1(\theta)$ violates (LL), then no other vector of incentive-compatible profits satisfies (LL).

Setting $\pi_s(\theta)=\pi_n(\theta)$ in (29), then plugging the resulting expression of $\pi_n(\theta)$, we see that (27) and (28) are jointly satisfied if and only if (35) is satisfied.

G Proof of Corollary 3

Using $\pi_s(\theta)=-L$ in (29), $\pi_n(\theta)$ is rewritten as

$$\pi_n(\theta)=\frac{L\sum_{s\neq 1,n}p_s(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_s'(\theta)}{p_s(\theta)}\right)-C_\theta\left(q^{fb}(\theta),\theta\right)}{p_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_n'(\theta)}{p_n(\theta)}\right)}.$$

Replacing $\sum_{s\neq 1,n}p_s(\theta)=1-(p_1(\theta)+p_n(\theta))$ and $\sum_{s\neq 1,n}p_s'(\theta)=-\left(p_1'(\theta)+p_n'(\theta)\right)$, $\pi_n(\theta)$ is further rewritten as $\pi_n^*(\theta)$. Recalling from (3) that $\pi_1(\theta)=-\sum_{s=2}^{n}\pi_s(\theta)\frac{p_s(\theta)}{p_1(\theta)}$ and using

$\pi_s(\theta) = -L$ together with $\pi_n^*(\theta)$ in the expression of $\pi_1(\theta)$, we obtain

$$\pi_1(\theta) = L\sum_{s\neq 1,n}\frac{p_s(\theta)}{p_1(\theta)} - \left(\frac{L\frac{p_1'(\theta)}{p_1(\theta)} - C_\theta\left(q^{fb}(\theta),\theta\right)}{p_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)} - L\right)\frac{p_n(\theta)}{p_1(\theta)}.$$

Replacing again $\sum_{s\neq 1,n} p_s(\theta) = 1 - (p_1(\theta) + p_n(\theta))$, $\pi_1(\theta)$ is is further rewritten as $\pi_1^*(\theta)$.

We are left with verifying that $\pi_1^*(\theta) \geq -L$ and $\pi_n^*(\theta) \geq -L$. The former is true because $p_n'(\theta) < 0$. The latter is implied by $\frac{p_1'(\theta)}{p_1(\theta)} > \frac{p_n'(\theta)}{p_n(\theta)}$ together with $C_\theta\left(q^{fb}(\theta),\theta\right)\frac{p_1(\theta)}{p_1'(\theta)} \leq L$, which is implied by (36) in turn.

H Proof of Corollary 4

Using (55) and $\frac{d\psi_{s,n}\left(\theta^+,\theta\right)}{d\theta^+} < 0$ (from the proof of Proposition 3), we see that $\Omega\left(\theta^+,\theta\right) < 0$, $\forall\theta^+ > \theta$, so that the term $\sum_{s\neq 1,n}\pi_s(\theta)\, p_s(\theta)\Omega\left(\theta^+,\theta\right)$ in the right-hand side of (32) increases as $\pi_s(\theta)$ is decreased. Hence, (32) is relaxed. Further using (55) and $\frac{d\psi_{s,n}\left(\theta^-,\theta\right)}{d\theta^-} < 0$ (from the proof of Proposition 3), we see that $\Omega\left(\theta^-,\theta\right) > 0$, $\forall\theta^- < \theta$, so that the term $\sum_{s\neq 1,n}\pi_s(\theta)\, p_s(\theta)\Omega\left(\theta^-,\theta\right)$ in the right-hand side of (31) increases as $\pi_s(\theta)$ is decreased. Hence, (31) is tightened.

I Proof of Corollary 5

(a) $\rho_n(\theta',\theta) > \rho_1(\theta',\theta)$ and $\rho_1(\theta',\theta) > \rho_s(\theta',\theta)$. It is immediate to see that (34) is satisfied. We next check situations where at least one of these inequalities is not satisfied.

(b) $\rho_n(\theta',\theta) > \rho_1(\theta',\theta)$ and $\rho_1(\theta',\theta) < \rho_s(\theta',\theta)$. Using these inequalities in (34) and rearranging, we obtain (39). Because property (1) implies that the right-hand side of (39) is above one, **(39)** is satisfied only if $\rho_n(\theta',\theta) - \rho_1(\theta',\theta) > \rho_s(\theta',\theta) - \rho_1(\theta',\theta) \Leftrightarrow \rho_n(\theta',\theta) > \rho_s(\theta',\theta)$.

(c) $\rho_n(\theta',\theta) < \rho_1(\theta',\theta)$ and $\rho_1(\theta',\theta) > \rho_s(\theta',\theta)$. Using these inequalities in (34) and rearranging, we obtain the converse of (39). Because property (1) implies that the right-hand side of (39) is above one, the converse of (39) is satisfied if $\rho_1(\theta',\theta) - \rho_n(\theta',\theta) < \rho_1(\theta',\theta) - \rho_s(\theta',\theta) \Leftrightarrow \rho_n(\theta',\theta) > \rho_s(\theta',\theta)$.

(d) $\rho_n(\theta',\theta) < \rho_1(\theta',\theta)$ and $\rho_1(\theta',\theta) < \rho_s(\theta',\theta)$. It is immediate to see that (34) is violated.

Taking $(a)-(d)$ altogether, (34) holds only if $\rho_s(\theta',\theta) < \max\{\rho_1(\theta',\theta), \rho_n(\theta',\theta)\}$. From (a) and (c), it is sufficient to have either $\rho_n(\theta',\theta) > \rho_1(\theta',\theta) > \rho_s(\theta',\theta)$ or $\rho_1(\theta',\theta) > \rho_n(\theta',\theta) > \rho_s(\theta',\theta)$, hence $\min\{\rho_1(\theta',\theta), \rho_n(\theta',\theta)\} > \rho_s(\theta',\theta)$. From (b), it is necessary and sufficient that (39) holds if $\rho_n(\theta',\theta) > \rho_s(\theta',\theta) > \rho_1(\theta',\theta)$. From (c), it is necessary and sufficient that the converse of (39) holds if $\rho_1(\theta',\theta) > \rho_s(\theta',\theta) > \rho_n(\theta',\theta)$. From (d), (34) is violated if $\rho_n(\theta',\theta) < \rho_1(\theta',\theta) < \rho_s(\theta',\theta)$.

J Proof of Corollary 6

Replacing $\pi_s(\theta) = -L$ in (33) and rearranging, (33) is rewritten as (40).

K Proof of Corollary 7

(37) implies (36) if and only if

$$\frac{\frac{\Delta C\left(\theta,\theta^-\right)}{\frac{p_n(\theta^-)}{p_n(\theta)}-\frac{p_1(\theta^-)}{p_1(\theta)}}-\frac{-\Delta C\left(\theta,\theta^+\right)}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}}{\sum_{s\neq 1,n} p_s(\theta)\left(\psi_{s,n}\left(\theta^-,\theta\right)-\psi_{s,n}\left(\theta^+,\theta\right)\right)} > \frac{\Delta C\left(\theta,\theta^-\right)p_1(\theta)}{p_1(\theta)-p_1(\theta^-)}.$$

Using the definition of $\psi_{s,n}\left(\theta',\theta\right)$ and grouping the terms that include $\Delta C\left(\theta,\theta^-\right)$, this becomes

$$\Delta C\left(\theta,\theta^-\right)\left[\frac{1}{\frac{p_n(\theta^-)}{p_n(\theta)}-\frac{p_1(\theta^-)}{p_1(\theta)}}+\frac{p_1(\theta)}{p_1(\theta)-p_1(\theta^-)}\left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)}\sum_{s\neq 1,n}p_s(\theta)-\sum_{s\neq 1,n}p_s(\theta^+)}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}\right.\right.$$
$$\left.\left.-\frac{\frac{p_1(\theta^-)}{p_1(\theta)}\sum_{s\neq 1,n}p_s(\theta)-\sum_{s\neq 1,n}p_s(\theta^-)}{\frac{p_1(\theta^-)}{p_1(\theta)}-\frac{p_n(\theta^-)}{p_n(\theta)}}\right)\right]$$
$$> \frac{-\Delta C\left(\theta,\theta^+\right)}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}$$

Using $\sum_{s\neq 1,n} p_s(\cdot) = 1 - p_1(\cdot) - p_n(\cdot)$ and rearranging further yields

$$\frac{\Delta C\left(\theta,\theta^-\right)p_1(\theta)}{p_1(\theta)-p_1(\theta^-)}\left(p_n(\theta)+\frac{1-p_n(\theta)-\frac{p_1(\theta)}{p_1(\theta^+)}\left(1-p_n(\theta^+)\right)}{\frac{p_1(\theta)}{p_1(\theta^+)}\left(\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}\right)}\right) > \frac{-\Delta C\left(\theta,\theta^+\right)}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}. \tag{60}$$

Take the expression in brackets in the left-hand side of (60) and factorize $p_n(\theta)$ to develop

$$p_n\left(\theta\right)\left(1+\frac{1-p_n(\theta)-\frac{p_1(\theta)}{p_1(\theta^+)}\left(1-p_n(\theta^+)\right)}{\frac{p_1(\theta)}{p_1(\theta^+)}\left(p_n\left(\theta\right)\frac{p_1(\theta^+)}{p_1(\theta)}-p_n(\theta^+)\right)}\right) = \frac{p_1(\theta^+)-p_1(\theta)}{p_1(\theta^+)-p_1(\theta)\frac{p_n(\theta^+)}{p_n(\theta)}}.$$

The converse of (35) is obtained by using this in (60) and rearranging.

L Proof of Lemma 5

We first conveniently reformulate the incentive constraints included in the reduced problem Γ', which we next use to write the Lagrangian. The proof of Lemma 5 follows thereafter.

Incentive constraints

(IC) for type θ_t and report $\theta_{t'}$ is specified as

$$R\left(\theta_t\right) \geq \pi_1\left(\theta_{t'}\right)p_1\left(\theta_t\right)+\pi_2\left(\theta_{t'}\right)p_2\left(\theta_t\right)+\pi_3\left(\theta_{t'}\right)p_3\left(\theta_t\right)+\Delta C\left(\theta_{t'},\theta_t\right). \tag{61}$$

Using the definition of $R(\theta_t)$, we write

$$\pi_1(\theta_t) = \frac{R(\theta_t)}{p_1(\theta_t)} - \pi_2(\theta_t)\frac{p_2(\theta_t)}{p_1(\theta_t)} - \pi_3(\theta_t)\frac{p_3(\theta_t)}{p_1(\theta_t)}. \tag{62}$$

Specifying (61) for type θ_1 and report θ_2, then using (62) for $\theta_t = \theta_2$ and rearranging, we obtain

$$\begin{aligned} & \pi_3(\theta_2)\,p_3(\theta_2) \\ \leq \; & \frac{R(\theta_1) - R(\theta_2)\frac{p_1(\theta_1)}{p_1(\theta_2)} - \Delta C(\theta_2,\theta_1) - \pi_2(\theta_2)\,p_2(\theta_2)\left(\frac{p_2(\theta_1)}{p_2(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}\right)}{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}}. \end{aligned} \tag{63}$$

Specifying (61) for type θ_3 and report θ_2, then using (62) for $\theta_t = \theta_2$ and rearranging, we further obtain

$$\begin{aligned} & \pi_3(\theta_2)\,p_3(\theta_2) \\ \geq \; & \frac{R(\theta_2)\frac{p_1(\theta_3)}{p_1(\theta_2)} - R(\theta_3) + \Delta C(\theta_2,\theta_3) - \pi_2(\theta_2)\,p_2(\theta_2)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_2(\theta_3)}{p_2(\theta_2)}\right)}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}. \end{aligned} \tag{64}$$

Using (62) for $\theta_t = \theta_3$ in (61) for type θ_2 and report θ_3 and rearranging yields

$$R(\theta_3) \leq \left[R(\theta_2) - \Delta C(\theta_3,\theta_2) - \sum_{s\neq 1}\pi_s(\theta_3)\,p_s(\theta_3)\left(\frac{p_s(\theta_2)}{p_s(\theta_3)} - \frac{p_1(\theta_2)}{p_1(\theta_3)}\right)\right]\frac{p_1(\theta_3)}{p_1(\theta_2)}. \tag{65}$$

Using the definition of $R(\theta_1)$, we write

$$\pi_3(\theta_1) = \frac{R(\theta_1)}{p_3(\theta_1)} - \pi_2(\theta_1)\frac{p_2(\theta_1)}{p_3(\theta_1)} - \pi_1(\theta_1)\frac{p_1(\theta_1)}{p_3(\theta_1)}. \tag{66}$$

Replacing in (61) as specified for type θ_2 and report θ_1, we find

$$R(\theta_1) \leq \left[R(\theta_2) - \Delta C(\theta_1,\theta_2) - \sum_{s\neq 3}\pi_s(\theta_1)\,p_s(\theta_1)\left(\frac{p_s(\theta_2)}{p_s(\theta_1)} - \frac{p_3(\theta_2)}{p_3(\theta_1)}\right)\right]\frac{p_3(\theta_1)}{p_3(\theta_2)}. \tag{67}$$

We rewrite Γ' as

$$\underset{\left\{q(\theta_t);\pi_{s\neq 3}(\theta_1);\pi_{s\neq 1}\left(\theta_{t\neq 1}\right),R(\theta_t)\right\}}{Max} \sum_{\theta_t}\left(S(q(\theta_t)) - C(q(\theta_t),\theta_t) - R(\theta_t)\right)f(\theta_t)$$

subject to

$$R(\theta_t) \geq 0,\;\; \pi_{s\neq 1}(\theta_2) \geq -L,\;\; (63),\;\; (64),\;\; (65),\;\; (67).$$

Lagrangian and optimality conditions

Denote ζ_t the multiplier associated with (PC) when type is θ_t; $\gamma_s(\theta_1)$ the multiplier associated with (LL) when signal is $s \neq 3$ and type is θ_1; $\gamma_s(\theta_3)$ the multiplier associated with (LL) when signal is $s \neq 1$ and type is θ_3; μ the multiplier associated with (63); λ the multiplier associated with (64); β the multiplier associated with (65); δ the multiplier associated with (67). The Lagrangian of Γ' is given by

$$
\begin{aligned}
& \mathcal{L}\left(q\left(\theta_t\right); \pi_{s\neq 3}\left(\theta_1\right); \pi_{s\neq 1}\left(\theta_2\right); \pi_{s\neq 1}\left(\theta_3\right); R\left(\theta_t\right)\right) \\
= & \sum_{\theta_t}\left(S\left(q\left(\theta_t\right)\right) - C\left(q\left(\theta_t\right), \theta_t\right) - R\left(\theta_t\right)\right) f\left(\theta_t\right) \\
& + \sum_{s\neq 3}\gamma_s\left(\theta_1\right)\left(\pi_s\left(\theta_1\right) + L\right) + \sum_{s\neq 1}\gamma_s\left(\theta_3\right)\left(\pi_s\left(\theta_3\right) + L\right) + \sum_{\theta_t}\zeta_t R\left(\theta_t\right) \\
& +\mu\left[\frac{R\left(\theta_1\right) - R\left(\theta_2\right)\frac{p_1(\theta_1)}{p_1(\theta_2)} - \Delta C\left(\theta_2, \theta_1\right) - \pi_2\left(\theta_2\right) p_2\left(\theta_2\right)\left(\frac{p_2(\theta_1)}{p_2(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}\right)}{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} - \pi_3\left(\theta_2\right) p_3(\theta_2)\right] \\
& +\lambda\left[\pi_3\left(\theta_2\right) p_3(\theta_2) + \frac{R\left(\theta_3\right) - R\left(\theta_2\right)\frac{p_1(\theta_3)}{p_1(\theta_2)} - \Delta C\left(\theta_2, \theta_3\right) + \pi_2\left(\theta_2\right) p_2\left(\theta_2\right)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_2(\theta_3)}{p_2(\theta_2)}\right)}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}\right] \\
& +\beta\left(\left[R\left(\theta_2\right) - \Delta C\left(\theta_3, \theta_2\right) - \sum_{s\neq 1}\pi_s\left(\theta_3\right) p_s\left(\theta_3\right)\left(\frac{p_s\left(\theta_2\right)}{p_s\left(\theta_3\right)} - \frac{p_1\left(\theta_2\right)}{p_1\left(\theta_3\right)}\right)\right]\frac{p_1\left(\theta_3\right)}{p_1\left(\theta_2\right)} - R\left(\theta_3\right)\right) \\
& +\delta\left(\left[R\left(\theta_2\right) - \Delta C\left(\theta_1, \theta_2\right) - \sum_{s\neq 3}\pi_s\left(\theta_1\right) p_s\left(\theta_1\right)\left(\frac{p_s\left(\theta_2\right)}{p_s\left(\theta_1\right)} - \frac{p_3\left(\theta_2\right)}{p_3\left(\theta_1\right)}\right)\right]\frac{p_3\left(\theta_1\right)}{p_3\left(\theta_2\right)} - R\left(\theta_1\right)\right).
\end{aligned}
$$

The first-order conditions with respect to $R\left(\theta_1\right)$, $R\left(\theta_2\right)$, $R\left(\theta_3\right)$, $\pi_2\left(\theta_2\right)$ and $\pi_3\left(\theta_2\right)$ are given by

$$
\begin{aligned}
(i) &: 0 = \zeta_1 - f\left(\theta_1\right) - \left(\delta - \frac{\mu}{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}}\right) \\
(ii) &: 0 = \zeta_2 - f\left(\theta_2\right) + \frac{p_3\left(\theta_1\right)}{p_3\left(\theta_2\right)}\delta - \frac{\frac{p_1(\theta_1)}{p_1(\theta_2)}\mu}{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} + \frac{p_1\left(\theta_3\right)}{p_1\left(\theta_2\right)}\left(\beta - \frac{\lambda}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}\right) \\
(iii) &: 0 = \zeta_3 - f\left(\theta_3\right) - \left(\beta - \frac{\lambda}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}\right) \\
(iv) &: 0 = \gamma_2\left(\theta_2\right) + p_2\left(\theta_2\right)\left(\lambda\frac{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_2(\theta_3)}{p_2(\theta_2)}}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}} - \mu\frac{\frac{p_2(\theta_1)}{p_2(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}}{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}}\right) \\
(v) &: 0 = \gamma_3\left(\theta_2\right) + p_3(\theta_2)\left(\lambda - \mu\right) \\
(vi) &: 0 = \gamma_s\left(\theta_3\right) - p_s\left(\theta_3\right)\left(\frac{p_s\left(\theta_2\right)}{p_s\left(\theta_3\right)} - \frac{p_1\left(\theta_2\right)}{p_1\left(\theta_3\right)}\right)\frac{p_1\left(\theta_3\right)}{p_1\left(\theta_2\right)}, \forall s\neq 1 \\
(vii) &: 0 = \gamma_s\left(\theta_1\right) - p_s\left(\theta_1\right)\left(\frac{p_s\left(\theta_2\right)}{p_s\left(\theta_1\right)} - \frac{p_3\left(\theta_2\right)}{p_3\left(\theta_1\right)}\right)\frac{p_3\left(\theta_1\right)}{p_3\left(\theta_2\right)}, \forall s\neq 3
\end{aligned}
$$

The first-order conditions with respect to $q(\theta_1)$, $q(\theta_2)$ and $q(\theta_3)$ are given by

$$\begin{aligned}
(viii) &: S'(q(\theta_1)) = C_q(q(\theta_1),\theta_1) + \frac{\delta}{f(\theta_1)} \frac{d\Delta C(\theta_1,\theta_2)}{dq(\theta_1)} \frac{p_3(\theta_1)}{p_3(\theta_2)} \\
(ix) &: S'(q(\theta_2)) = C_q(q(\theta_2),\theta_2) + \frac{\mu}{f(\theta_2)} \frac{\frac{d\Delta C(\theta_2,\theta_1)}{dq(\theta_2)}}{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} - \frac{\lambda}{f(\theta_2)} \frac{\frac{-d\Delta C(\theta_2,\theta_3)}{dq(\theta_2)}}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}} \\
(x) &: S'(q(\theta_3)) = C_q(q(\theta_3),\theta_3) + \frac{\beta}{f(\theta_3)} \frac{d\Delta C(\theta_3,\theta_2)}{dq(\theta_3)} \frac{p_1(\theta_3)}{p_1(\theta_2)}
\end{aligned}$$

Proof of case (1) **and case** (2)

From (v) we see that either $\gamma_3(\theta_2) > 0$ and $\lambda < \mu$, or $\gamma_3(\theta_2) = 0$ and $\lambda = \mu$. It follows that $\lambda \leq \mu$. Using this inequality in (iv), together with (22), it follows that $\gamma_2(\theta_2) > 0$. Hence, the solution is such that either $\gamma_3(\theta_2) > 0$, $\gamma_2(\theta_2) > 0$ and $\lambda < \mu$, or $\gamma_3(\theta_2) = 0$, $\gamma_2(\theta_2) > 0$ and $\lambda = \mu$. Accordingly, we have case (1), where $\pi_2(\theta_2) = \pi_3(\theta_2) = -L$; and case (2), where $\pi_2(\theta_2) = -L < \pi_3(\theta_2)$. Because $\lambda < \mu$ in case (1), there is no conflict between (64) and (65). If (16) and (17) were satisfied, then first best would be implemented. Because first best is not implemented, case (1) exists only if (16) and (17) are violated. Because $\lambda = \mu$ in case (2), (64) and (65) are both binding. If (23) were satisfied, then we know from the first-best analysis that (64) and (65) would both be slack. But this contradicts the result that these constraints are both binding. Hence, case (2) exists only if (23) is violated.

We are left with assessing how $\pi_s(\theta_1)$, $\forall s \neq 3$, and $\pi_s(\theta_3)$, $\forall s \neq 1$, are set. From (vi) and (vii) we see that $\gamma_s(\theta_1) > 0$, $\forall s \neq 3$, and $\gamma_s(\theta_3) > 0$, $\forall s \neq 1$. Hence, at optimum, $\pi_1(\theta_1) = \pi_2(\theta_1) = -L$ and $\pi_2(\theta_3) = \pi_3(\theta_3) = -L$.

M Proof of Proposition 4

Because (17) and (16) are violated, case (1) of Lemma 5 applies. Recall from the proof of Lemma 5 that $\gamma_3(\theta_2) > 0$, $\gamma_2(\theta_2) > 0$ and $\lambda < \mu$. Because $\lambda < \mu$, $\mu > 0$ and (63) is binding. Because $\gamma_3(\theta_2) > 0$ and $\gamma_2(\theta_2) > 0$, $\pi_2(\theta_2) = \pi_3(\theta_2) = -L$. Replacing in (63) we obtain

$$R(\theta_1) = R(\theta_2) \frac{p_1(\theta_1)}{p_1(\theta_2)} + \Delta C(\theta_2,\theta_1) - L \frac{p_1(\theta_2) - p_1(\theta_1)}{p_1(\theta_2)}. \quad (68)$$

We now show that $\delta = 0$, which entails that (67) is slack. Replacing $\pi_s(\theta_1) = -L$, $\forall s \neq 3$, from Lemma 5, (67) is rewritten as

$$R(\theta_1) \leq R(\theta_2) \frac{p_3(\theta_1)}{p_3(\theta_2)} - \Delta C(\theta_1,\theta_2) \frac{p_3(\theta_1)}{p_3(\theta_2)} + L \frac{p_3(\theta_1) - p_3(\theta_2)}{p_3(\theta_2)}. \quad (69)$$

Replacing (68), (69) further becomes

$$R(\theta_2) \left(\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)} \right) + L \left(\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)} \right) \geq \Delta C(\theta_2,\theta_1) + \Delta C(\theta_1,\theta_2) \frac{p_3(\theta_1)}{p_3(\theta_2)}.$$

Because $R(\theta_2) \geq 0$ and $\frac{p_3(\theta_1)}{p_3(\theta_2)} > \frac{p_1(\theta_1)}{p_1(\theta_2)}$, this is strictly satisfied if

$$\frac{-\Delta C(\theta_1, \theta_2)}{\Delta C(\theta_2, \theta_1)} > \frac{p_3(\theta_2)}{p_3(\theta_1)}.$$

Knowing that $C_{q\theta}(\cdot, \cdot) > 0$ and that $p_3'(\cdot) < 0$, the above inequality holds if $q(\theta_1) > q(\theta_2)$ at the solution. Because this is the case, indeed, we can conclude that $\delta = 0$.

Replacing $\pi_s(\theta_1) = -L$, $\forall s \neq 3$, from Lemma (5), (65) is rewritten as

$$R(\theta_3) \leq R(\theta_2) \frac{p_1(\theta_3)}{p_1(\theta_2)} - \Delta C(\theta_3, \theta_2) \frac{p_1(\theta_3)}{p_1(\theta_2)} + L \frac{p_1(\theta_3) - p_1(\theta_2)}{p_1(\theta_2)}. \tag{70}$$

Moreover, recalling that $\pi_s(\theta_2) = -L$, $\forall s \neq 1$, (64) is rewritten as

$$R(\theta_3) \geq R(\theta_2) \frac{p_1(\theta_3)}{p_1(\theta_2)} + \Delta C(\theta_2, \theta_3) + L \frac{p_1(\theta_3) - p_1(\theta_2)}{p_1(\theta_2)}. \tag{71}$$

Taking together (70) and (71), we find (43).

We now show that $\beta > 0$. Take first $\beta < \frac{\lambda}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}$. It follows from (ii) and $\delta = 0$ that $\zeta_2 > 0$, hence $R(\theta_2) = 0$. Replacing $R(\theta_2) = R(\theta_3) = 0$, (70) is rewritten as

$$\Delta C(\theta_3, \theta_2) \frac{p_1(\theta_3)}{p_1(\theta_3) - p_1(\theta_2)} \leq L.$$

Suppose that $\beta = 0$ and replace in (x). We see that $q(\theta_3) = q^{fb}(\theta_3)$, and the above condition coincides with (17) as specified for $\theta_{t'} = \theta_2$. This contradicts the assumption that (17) is violated. Hence, if $\beta < \frac{\lambda}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}$, it must be $\beta > 0$. Take now $\beta \geq \frac{\lambda}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}$. If $\beta = 0$, then it must be $\lambda = 0$. Replacing $\beta = \lambda = 0$ in (ii), again we find that $\zeta_2 > 0$ and we end up with a contradiction. Hence, if $\beta \geq \frac{\lambda}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}$, then again it must be $\beta > 0$; hence, (70) is binding. Replacing $R(\theta_3) = 0$ in (70), which is binding, we derive (42). Using (42) in (68), we further derive (41).

N Proof of Remark 1

Recall from the proof of Proposition 4 that (70) and (71) imply (43).

Take first $\beta \geq \frac{\lambda}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}$. It follows from (iii) that $\zeta_3 > 0$, hence $R(\theta_3) = 0$. Take next $\beta < \frac{\lambda}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}$. From (ii) and $\delta = 0$ it follows that $\zeta_2 > 0$, hence $R(\theta_2) = 0$. Moreover, because $\beta < \frac{\lambda}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}$, it must be $\lambda > 0$; hence, (71) is binding. Replacing $R(\theta_2) = 0$ and $\pi_{s \neq 1}(\theta_2) = -L$, (71) is rewritten as

$$R(\theta_3) = \Delta C(\theta_2, \theta_3) + L \frac{p_1(\theta_3) - p_1(\theta_2)}{p_1(\theta_2)}. \tag{72}$$

Suppose that $R(\theta_3) > 0$. From (72) it is necessary to have

$$-\Delta C(\theta_2, \theta_3) \frac{p_1(\theta_2)}{p_1(\theta_3) - p_1(\theta_2)} < L. \tag{73}$$

Recall that $R(\theta_2) = 0$ and replace in (68). We see that $R(\theta_1) \geq 0$ if and only if

$$\Delta C(\theta_2, \theta_1) \frac{p_1(\theta_2)}{p_1(\theta_2) - p_1(\theta_1)} \geq L. \tag{74}$$

From (73) and (74) it is further necessary that (44) be violated. Hence, if (44) is satisfied, the hypothesis that $R(\theta_3) > 0$ leads to a contradiction.

O Proof of Proposition 5

Because (23) is violated whereas (17) and (16) are satisfied, case (2) of Lemma 5 applies. Recall from the proof of Lemma 5 that $\gamma_3(\theta_2) = 0$, $\gamma_2(\theta_2) > 0$ and $\lambda = \mu$. Because $\lambda = \mu > 0$, (63) and (64) are both binding. Because $\gamma_3(\theta_2) = 0 < \gamma_2(\theta_2)$, it is $\pi_3(\theta_2) > -L = \pi_2(\theta_2)$ at optimum.

Given that $\lambda = \mu > 0$, one has $\beta = \delta = 0$. It follows from (ii) that $\zeta_2 > 0$, hence $R(\theta_2) = 0$. Moreover, (i) and (iii) are respectively rewritten as

$$\begin{aligned} (i') &: \ 0 = \zeta_1 - f(\theta_1) + \frac{\mu}{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} \\ (iii') &: \ 0 = \zeta_3 - f(\theta_3) + \frac{\mu}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}. \end{aligned}$$

Take $\frac{f(\theta_3)}{f(\theta_1)} = \chi$. Replacing $f(\theta_3) = \chi f(\theta_1)$ in (iii'), we obtain

$$f(\theta_1) = \frac{\zeta_3}{\chi} + \frac{\mu/\chi}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}.$$

Using this in (i'), we further obtain $\chi\zeta_1 = \zeta_3$. Hence, either $\zeta_1 = \zeta_3 = 0$, or $\zeta_1 > 0$ and $\zeta_3 > 0$. We found that $\zeta_2 > 0$. Hence, if $\zeta_1 > 0$ and $\zeta_3 > 0$, then first best is implemented, which leads to a contradiction. Therefore, $\zeta_1 = \zeta_3 = 0$.

Take $\frac{f(\theta_3)}{f(\theta_1)} > \chi$. Using $f(\theta_3) > \chi f(\theta_1)$ in (iii'), we obtain

$$\frac{\zeta_3}{\chi} + \frac{\mu/\chi}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}} > f(\theta_1) \tag{75}$$

Using (75) in (i'), we see that $\zeta_1 < (\zeta_3/\chi)$. Hence, $\zeta_3 > 0$ and so $R(\theta_3) = 0$.

Take $\frac{f(\theta_3)}{f(\theta_1)} < \chi$. Using $f(\theta_3) < \chi f(\theta_1)$ in (iii'), we obtain the converse of (75). Using it in (i'), we see that $\zeta_1 > (\zeta_3/\chi)$. Hence, $\zeta_1 > 0$ and so $R(\theta_1) = 0$.

Replacing $R(\theta_2)=0$ and $\pi_2(\theta_2)=-L$ in (63) and (64), which are binding, we obtain

$$R(\theta_1) = \Delta C(\theta_2,\theta_1) - Lp_2(\theta_2)\left(\frac{p_2(\theta_1)}{p_2(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}\right) + \pi_3(\theta_2)\,p_3(\theta_2)\left(\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}\right) \quad (76)$$

$$R(\theta_3) = \Delta C(\theta_2,\theta_3) + Lp_2(\theta_2)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_2(\theta_3)}{p_2(\theta_2)}\right) - \pi_3(\theta_2)\,p_3(\theta_2)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}\right) \quad (77)$$

If $\frac{f(\theta_3)}{f(\theta_1)} > \chi$, so that $R(\theta_3)=0$, then from (77) we obtain

$$\pi_3(\theta_2) = \frac{L\frac{p_1(\theta_3)-p_1(\theta_2)}{p_1(\theta_2)} + \Delta C(\theta_2,\theta_3)}{p_3(\theta_2)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}\right)} - L$$

Replacing in (76), we find $R(\theta_1) = R_1^{sb}$. If $\frac{f(\theta_3)}{f(\theta_1)} < \chi$, so that $R(\theta_1)=0$, then from (76) we obtain

$$\pi_3(\theta_2) = \frac{L\frac{p_1(\theta_2)-p_1(\theta_1)}{p_1(\theta_2)} - \Delta C(\theta_2,\theta_1)}{p_3(\theta_2)\left(\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}\right)} - L.$$

Replacing in (77), we find $R(\theta_3) = R_3^{sb}$. If $\frac{f(\theta_3)}{f(\theta_1)} = \chi$, then by rewriting (77) for $\pi_3(\theta_2)$ we obtain

$$\pi_3(\theta_2) = \frac{\Delta C(\theta_2,\theta_3) + Lp_2(\theta_2)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_2(\theta_3)}{p_2(\theta_2)}\right)}{p_3(\theta_2)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}\right)} - \frac{R(\theta_3)}{p_3(\theta_2)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}\right)}.$$

Replacing in (76), we find $R(\theta_1) = R_1^{sb} - \chi R(\theta_3)$.

P Proof of Corollary 8

The sufficiency of (45) to have $R(\theta_2)=0$ is immediate from Remark 1. To show the sufficiency of (45) for the incentive constraints omitted in Γ' to be satisfied, we consider the two cases of Lemma 5 one by one.

Case (1)

Recall from Lemma 5 that $\pi_2(\theta_3)=\pi_3(\theta_3)=-L$. With this, (IC) for type θ_1 and report θ_3 is specified as

$$R(\theta_1) \geq R(\theta_3)\frac{p_1(\theta_1)}{p_1(\theta_3)} + \Delta C(\theta_3,\theta_1) - L\frac{p_1(\theta_3)-p_1(\theta_1)}{p_1(\theta_3)}. \quad (78)$$

Recall from the proof of Proposition 4 that $R(\theta_1)$ is given by (68). Replacing in (78) together with $R(\theta_3)$ and rearranging, we obtain

$$R(\theta_2) \geq (\Delta C(\theta_3,\theta_1) - \Delta C(\theta_2,\theta_1))\frac{p_1(\theta_2)}{p_1(\theta_1)} - L\frac{p_1(\theta_3)-p_1(\theta_2)}{p_1(\theta_3)},$$

which holds for any $R(\theta_2) \geq 0$ because $q(\theta_2) \geq q(\theta_3)$ at optimum so that $\Delta C(\theta_3, \theta_1) \leq \Delta C(\theta_2, \theta_1)$.

(IC) for type θ_3 and report θ_1 is specified as

$$R(\theta_3) \geq \pi_1(\theta_1) p_1(\theta_3) + \pi_2(\theta_1) p_2(\theta_3) + \pi_3(\theta_1) p_3(\theta_3) + \Delta C(\theta_1, \theta_3).$$

Recalling from Lemma 5 that $\pi_2(\theta_3) = \pi_3(\theta_3) = -L$, this is rewritten as

$$R(\theta_3) \geq R(\theta_1) \frac{p_3(\theta_3)}{p_3(\theta_1)} + \Delta C(\theta_1, \theta_3) - L \frac{p_3(\theta_1) - p_3(\theta_3)}{p_3(\theta_1)}. \tag{79}$$

Replacing (68), we obtain

$$\begin{aligned} R(\theta_3) \geq\ & R(\theta_2) \frac{p_1(\theta_1)}{p_1(\theta_2)} \frac{p_3(\theta_3)}{p_3(\theta_1)} \\ & + \Delta C(\theta_2, \theta_1) \frac{p_3(\theta_3)}{p_3(\theta_1)} + \Delta C(\theta_1, \theta_3) - L \frac{p_3(\theta_3)}{p_3(\theta_1)} \left(\frac{p_3(\theta_1)}{p_3(\theta_3)} - \frac{p_1(\theta_1)}{p_1(\theta_2)} \right). \end{aligned}$$

This is implied by (71) if and only if

$$\begin{aligned} & R(\theta_2) \frac{p_1(\theta_1)}{p_1(\theta_2)} \left(\frac{p_1(\theta_3)}{p_1(\theta_1)} - \frac{p_3(\theta_3)}{p_3(\theta_1)} \right) \\ \geq\ & \Delta C(\theta_2, \theta_1) \frac{p_3(\theta_3)}{p_3(\theta_1)} - \Delta C(\theta_2, \theta_3) + \Delta C(\theta_1, \theta_3) - L \frac{p_1(\theta_1)}{p_1(\theta_2)}. \end{aligned} \tag{80}$$

Rewrite the right-hand side as

$$-\Delta C(\theta_2, \theta_1) \left(\frac{\Delta C(\theta_2, \theta_3) - \Delta C(\theta_1, \theta_3)}{\Delta C(\theta_2, \theta_1)} - \frac{p_3(\theta_3)}{p_3(\theta_1)} \right) - L \frac{p_1(\theta_1)}{p_1(\theta_2)}$$

and observe that $\frac{\Delta C(\theta_2, \theta_3) - \Delta C(\theta_1, \theta_3)}{\Delta C(\theta_2, \theta_1)} \geq 1$ because

$$C(q(\theta_1), \theta_3) - C(q(\theta_1), \theta_1) \geq C(q(\theta_2), \theta_3) - C(q(\theta_2), \theta_1),$$

which is true provided $C_{\theta q}(\cdot, \cdot) > 0$. Further observe that $\frac{p_3(\theta_3)}{p_3(\theta_1)} < 1$ under property (1). Hence, (80) is satisfied $\forall R(\theta_2) \geq 0$.

Case (2)

Recall that (IC) for type θ_1 and report θ_3 is specified as (78). From the proof of Proposition (5), $R(\theta_1) = R_1^{sb} - \chi R(\theta_3)$. Using this equality in (78), we obtain

$$R_1^{sb} \geq R(\theta_3) \left(\frac{p_1(\theta_1)}{p_1(\theta_3)} + \chi \right) + \Delta C(\theta_3, \theta_1) - L \frac{p_1(\theta_3) - p_1(\theta_1)}{p_1(\theta_3)}$$

Because $R_1^{sb} > 0$, this is satisfied if

$$R(\theta_3) \leq \frac{-\Delta C(\theta_3, \theta_1) + L\frac{p_1(\theta_3) - p_1(\theta_1)}{p_1(\theta_3)}}{\frac{p_1(\theta_1)}{p_1(\theta_3)} + \chi}.$$

This is tightest for $R(\theta_3) = R_3^{sb}$. Replacing the expression of R_3^{sb} and rearranging, it becomes

$$\Psi_{1,3}(\theta_1, \theta_2, \theta_3) \leq \frac{-\Delta C(\theta_3, \theta_1) + L\frac{p_1(\theta_3) - p_1(\theta_1)}{p_1(\theta_3)}}{\left(\frac{p_1(\theta_1)}{p_1(\theta_3)} + \chi\right)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}\right) p_2(\theta_2)\Delta\psi} + L$$

Observing that the right-hand side is positive and using the definition of $\Psi_{1,3}(\theta_1, \theta_2, \theta_3)$, we see that $\exists \phi > 0$ such that if (45) is satisfied, then the incentive constraint is satisfied.

Recall that (IC) for type θ_3 and report θ_1 is given by (79). Using again $R(\theta_1) = R_1^{sb} - \chi R(\theta_3)$, (79) is rewritten as

$$R(\theta_3) \geq \left(R_1^{sb} - \chi R(\theta_3)\right)\frac{p_3(\theta_3)}{p_3(\theta_1)} + \Delta C(\theta_1, \theta_3) - L\frac{p_3(\theta_1) - p_3(\theta_3)}{p_3(\theta_1)}.$$

This is tightest for $R(\theta_3) = 0$. Replacing $R(\theta_3) = 0$ and the expression of R_1^{sb} and then rearranging, we rewrite

$$\Psi_{1,3}(\theta_1, \theta_2, \theta_3) \leq \frac{-\Delta C(\theta_1, \theta_3) + L\frac{p_3(\theta_1) - p_3(\theta_3)}{p_3(\theta_1)}}{\frac{p_3(\theta_3)}{p_3(\theta_1)}\left(\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}\right) p_2(\theta_2)\Delta\psi} + L.$$

The right-hand side is positive. Hence, using the definition of $\Psi_{1,3}(\theta_1, \theta_2, \theta_3)$, we see that $\exists \phi > 0$ such that if (45) is satisfied, then the incentive constraint is satisfied.