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Extremism Drives Out Moderation

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Abstract

This article investigates the impact of the distribution of preferences on equilibrium behavior in conflicts that are modeled as all-pay auctions with identity-dependent externalities. In this context, we define centrists and radicals using a willingness-to-pay criterion that admits preferences more general than a simple ordering on the line. Through a series of examples, we show that substituting the auction contest success function for the lottery contest success function in a conflict may alter the relative expenditures of centrists and radicals in equilibrium. Extremism, characterized by a higher per capita expenditure by radicals than centrists, may persist and lead to a higher aggregate expenditure by radicals, even when they are relatively small in number. Moreover, we show that centrists may in the aggregate expend zero, even if they vastly outnumber radicals. Our results demonstrate the importance of the choice of the institutions of conflict, as modeled by the contest success function, in determining the role of extremism and moderation in economic, political, and social environments.

JEL-Code: D720, D740, C720, D440.

Keywords: conflict, all-pay auction, identity-dependent externalities, radicalism, extremism, contest success function.

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1 Introduction

It is axiomatic that the nature of conflict depends on the institutions of conflict. In this paper we examine conflicts in which economic agents expend scarce resources in order to achieve their preferred outcome among a set of alternatives. If an agent secures his preferred alternative we say that the agent "wins." Otherwise, the agent "loses." In this respect the conflicts that we examine are contests as defined, say, in Konrad (2009). Our approach differs from much of the literature on contests in that agents are not indifferent to the identity of the winning agent in the event that they themselves lose. That is, we examine contests with identity-dependent externalities.

In much of the theoretical work on conflict to date the institutions of conflict have been black-boxed by the application of a contest success function - a function that maps the vector of agents' resource expenditures in the conflict into their respective probabilities of winning their preferred outcome. Two prominent types of contest success functions (henceforth, CSFs) employed in the literature are the "lottery" CSF (Tullock, 1980), in which the probability that an agent wins his preferred outcome equals the ratio of the agent's expenditure to the sum of all agents' expenditures, and the "auction" CSF, in which the agent with the greatest expenditure wins his preferred outcome with certainty. The lottery CSF is a popular method of modeling conflicts in which the outcome is determined not just by the respective expenditures of resources, but also a substantial random component. An auction CSF may be viewed as approximating environments in which random exogenous factors play little role in influencing the outcome of the conflict. Because of the discontinuity in the auction CSF when agents are tied for the highest expenditure, small differences in (positive) expenditure may lead to large differences in the probability of winning. That is, in contests, the auction CSF represents cutthroat competition in sunk expenditure, much the way that classical Bertrand competition is cutthroat competition in price. With the lottery

CSF competition is softened by randomness in the outcome, conditional on the profile of expenditures.

Contests with identity-dependent externalities utilizing a lottery CSF have been examined by Linster (1993) and Esteban and Ray (1999). Linster (1993) demonstrates that with a constant unit cost of expenditure, pure strategy Nash equilibrium profiles of expenditures may be obtained as the solution to a nonlinear system of equations.[1] He analyzes two three-player environments in more detail, including a comparative statics analysis that links total conflict and social surplus to the extent of the externalities. Esteban and Ray (1999) extend Linster's (1993) model by considering groups of agents, with heterogeneous preferences across groups but homogeneous preferences within each group. Each agent has an identical strictly convex cost of expenditure function, and free-rider problems are assumed away by requiring that each group of agents acts as a single agent with the group's aggregate cost of expenditure function (and dividing the resulting expenditure equally). Hence, larger groups have lower costs. The current contribution reexamines several of the issues addressed in these papers applying the auction CSF. That is, we examine all-pay auctions with identity-dependent externalities.

To the best of our knowledge, we are the first to study equilibria of the all-pay auction with identity-dependent externalities.[2] In this sense we provide a bridge between models of conflict such as Linster (1993) and Esteban and Ray (1999) that utilize a lottery CSF and the growing literature on winner-pay auctions with identity-dependent externalities in which agents place bids, an auction CSF is employed, but generally all bids except for the winner's are refunded. Jehiel and Moldovanu(2006) review this literature and note

[1]Linster (1993) argues that such a solution exists unless the contest is degenerate in the sense that players are indifferent to the outcome.

[2]Konrad(2006) examines the effect of silent shareholdings in an all-pay auction framework with complete information and finds that the social value may increase or decrease depending on the identity of the firm that holds a share in its competitor. However, Konrad does not further analyze settings in which three firms are active in equilibrium and allows only one player's valuation to be endogenous.

that the endogeneity of valuations in winner-pay auctions is the main driving force behind many new, and interesting phenomena that arise even in complete information settings.[3] A comprehensive treatment of the (first-price) winner-pay auction with identity-dependent externalities and complete information appears in Funk (1996).

As noted by Esteban and Ray, identity-dependent externalities can, under certain conditions, impart a natural "metric" measuring the distance between players.[4] If, for every $i \in I = \{1, 2, \ldots, n\}$, $v_i = (v_{i1}, v_{i2}, \ldots, v_{in})$ is the vector of payoffs received by player i when players $1, 2, \ldots, n$, respectively, win their preferred option, it is natural to extend the definition of "reach" due to Siegel (2009) to account for identity-dependent externalities. More precisely, let $r_{ij} = v_{ii} - v_{ij}$ be player i's reach with respect to player j. That is, r_{ij} is the maximum amount that player i would be willing to expend in order to win with certainty rather than have player j win with certainty. Under the assumption of symmetry, $r_{ij} = r_{ji}$, players' reaches may be viewed as a distance between the preferred outcomes of players based upon the players' willingness to outbid each other to achieve their most favored outcome. Players have similar preferences over their preferred outcomes if they value the success of the other in terms similar to their own; that is, if $r_{ij} = r_{ji}$ is small. This notion of the distance between players' preferred outcomes allows the quantification of the terms radical and centrist in terms of the set of reaches over all player pairs. Player i is a *radical* if he is an outlier in the sense that he is a player that attains the highest reach, $\max\{r_{ij} | i, j \in I\}$, among all player pairs, with the additional qualification that any player pair k, l attaining the same reach must include player i. A player who is not a radical is a *centrist*.

[3] For instance, in first-price winner-pay auctions, Funk (1996) and Jehiel and Moldovanu (1996) show that multiple payoff nonequivalent equilibria may arise. Jehiel and Moldovanu (1996) show that if players can commit in a pre-auction stage not to participate, both potential winners and losers may choose non-participation, despite the inability to avoid the negative externality. Janssen and Moldovanu (2004) show that revenue and efficiency may be unrelated to each other.

[4] Esteban and Ray (1999) do not show formally that the distance measure induced by preferences over outcomes is a metric. See section 2 for our assumptions under which there exists a semi-metric induced by players' willingness to outbid each other.

In the sections that follow we incorporate the distributions of players' preferences, as summarized by their reaches, in all-pay auctions with identity-dependent externalities to examine the behavior of centrists and radicals in Nash equilibrium. As in Esteban and Ray (1999) *extremism* refers to environments in which equilibrium expected per capita expenditures are higher for radical players than for centrists. *Moderation* refers to environments in which this ranking is reversed. We analyze simple scenarios similar to those in Esteban and Ray (1999) and find that expected per capita expenditures are higher for radicals than for centrists. This advantage may lead to a higher aggregate expected expenditure by radicals, even when they are relatively small in number. In fact, centrists may in the aggregate expend zero with certainty, even when they vastly outnumber radicals. Thus, *extremism drives out moderation* if an auction rather than a lottery CSF is employed.

Our findings are in the spirit of Osborne et al. (2000), who show that players representing central positions will not participate in meetings when there is an identical fixed cost of participation and the outcome is a compromise between the participants. We similarly find the non-participation of centrists in environments that are different from those in Osborne et al. (2000) in two fundamental ways. First, in our model, players' expenditures are variable and influence the outcome of the conflict. Second, the outcome of the conflict is the position of the player with the highest expenditure.

In the next section we provide a model of the all-pay auction with identity-dependent externalities and define players' proximity based on their preferences. We then analyze equilibrium behavior in different three-player environments. In Section 3 we conclude with a brief discussion of welfare, contests between groups, and more general assumptions on the cost of bidding.

2 The Model

We examine all-pay auctions with identity-dependent externalities under complete information. In an all-pay auction all players place their bids simultaneously, the player with the highest bid wins the prize, and all players pay their bid. In order to capture the idea that a player is not indifferent to who wins the prize if he does not, we represent a player's valuation of the outcome as an n-dimensional vector rather than a scalar. Each player's valuation of the outcome is a vector $v_i = (v_{i1}, v_{i2}, \dots, v_{in})$, $i \in I = \{1, \dots, n\}$, where v_{ij} is the value to player i if player j wins the prize. Externalities are not restricted to being positive or negative only, but we assume that players strictly prefer to win the prize.

Assumption 1. $\forall i \in I : v_{ii} > v_{ij} \forall j \in I, j \neq i$.

Given a profile of bids, $b = (b_1, \dots, b_n)$, player i's payoff is

$$u_i(b) = \sum_{j \in I} p_j(b) v_{ij} - C(b_i),$$

where $p_j : \mathbb{R}_+^n \to [0,1]$ is player j's probability of winning given the profile of bids, and $C : \mathbb{R}_+ \to \mathbb{R}_+$ is the player's cost, which only depends on his own bid. With an auction CSF

$$p_j(b) = \begin{cases} 0 & \text{if } \exists k \in I : b_k > b_j \\ 1 & \text{if } b_j > b_k \forall k \neq j \\ \frac{1}{m} & \text{if } j \text{ ties with } m-1 \text{ other players for the high bid} \end{cases} .$$

For simplicity we assume in the following analysis that $C(b) = b$, i.e. players have constant unit marginal cost. However, our results hold generally if players have identical, continuous, strictly increasing, and unbounded cost functions with $C(0) = 0$.[5] We expand a player's expected utility, $u_i(\cdot)$, and probabilty of winning, $p_i(\cdot)$, to the domain of mixed strategies.

[5]In this case we can employ our analysis to a transformed bid , $\beta = C(b)$. We elaborate on other potential assumptions on cost in section 3.

A mixed strategy $F_i : \mathbb{R}_+ \to [0,1]$ of player $i \in I$ is a cumulative probability distribution function over his bids. If a player bids zero with probability one, we refer to this strategy as *staying out* of the conflict. When a player submits a positive bid with strictly positive probability we say that he *actively participates* in the conflict.

We aim to analyze the effects that the distribution of preferences has on strategic behavior in all-pay auctions and for this purpose focus on three-player environments ($I = \{1,2,3\}$). For these environments we define radicalism and centrism based on the profile of players' valuations. More precisely, let $r_{ij} := v_{ii} - v_{ij}$, $i, j \in I$, be player i's *reach*[6] with respect to player j, meaning that r_{ij} is the maximum player i would be willing to bid in order to outbid player j, if players i and j were the only actively competing players. To ensure an unambiguous measure of preference proximity we assume the following.

Assumption 2. *Inter-agent antagonism is symmetric, i.e.* $r_{ij} = r_{ji}\ \forall i, j \in I$.

Under assumptions 1 and 2, $d(i,j) := r_{ij}$ can be interpreted as a distance between players that reflects preferences over outcomes in the sense that player i (weakly) prefers the outcome where j wins over the outcome where k wins if and only if $d(i,j) \leq d(i,k)$, $i, j, k \in I$. In fact, $d(i,j) := r_{ij}$ has the properties of a semi-metric:

1. Non-Negativity: $d(i,j) := r_{ij} = v_{ii} - v_{ij} \geq 0$ by assumption 1,

2. Identity of Indiscernibles: $d(i,j) := r_{ij} = 0$ if and only if $i = j$ also by assumption 1,

3. Symmetry: $d(i,j) := r_{ij} = r_{ji} = d(j,i)$ by assumption 2.

Note, however, that we do not restrict our analysis to environments in which the triangle inequality holds, as there is no intuitive motivation for this property in the context. Therefore, $d(i,j) := r_{ij}$ need not be a metric.

[6]This definition is based on Siegel(2009) but accounts for the identity-dependent externalities.

For example, $d(i,j) = r_{ij}$ could be generated by a spatial preference model in which players engage in an all-pay auction to implement their distinct ideal points in a finite dimensional real issue space. Suppose players possess identical, additively separable utility functions in which the player's bid is subtracted from a subutility function decreasing in the Euclidean distance between the player's ideal point and the implemented ideal point. Whether or not the triangle inequality may be violated rests on the curvature of the identical subutility functions. If these functions are strictly concave in the Euclidean distance (reflecting increasing marginal disutility in Euclidean distance), then the triangle inequality may be violated. If the subutility functions are linear or convex in the Euclidean distance between ideal points, then the triangle inequality holds.

Given this framework, we define the players' distribution of preferences based on their willingness to outbid others as well as other players' reciprocal antagonism towards them. Intuitively, a player who favors a radical outcome will generally face stronger opposition from his rivals, and in turn be willing to expend high effort to support it.

Definition 1. A player $i \in I$ is called *radical*,[7] if

$$i \in \bigcap_{r_{st}=\max\{r_{ij}|i,j\in I\}} \{s,t\}$$

Definition 2. A player $i \in I$ is called *centrist*, if i is not radical.

According to Definition 1 we call a player i radical if he is willing to bid up to the maximum of all reaches when competing with some other player, and additionally other players would not be willing to bid that high unless possibly if they were competing with i.

Following Esteban and Ray(1999), we refer to *extremism* as a situation where all radical players expend in expectation more effort per capita than do centrists in order to reach their

[7]To our knowledge no formal definition of a *radical* in an n-player environment exists in the literature. However, in the symmetric three-player environments considered in this article many possible definitions would lead to exactly the same classification of players as definition 1 above.

preferred outcome. Alternatively, a situation in which centrists expend more effort would be referred to as *moderation*. In the following paragraphs we separately consider the cases of (A) two radical players and one centrist, (B) one radical player and two centrists, and (C) the all-pay auction without radical players.

2.1 Two Radicals

Let players 1 and 3 be radical and player 2 be the centrist. Without loss of generality we consider the case where $d(2,1) = d(2,3)$. Figure 1 illustrates the ranking of the r_{ij}'s in this case. We refer to this all-pay auction as Γ_{21}, where 2 refers to the number of radicals and 1 to the lone centrist.

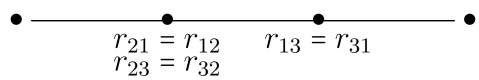

Figure 1: The case of two radical players and one centrist.

We find that in any Nash equilibrium of Γ_{21} both radicals will actively participate. Moreover, the Nash equilibrium of Γ_{21} is unique and symmetric. It has the property that the centrist player stays out of the conflict. This stands in contrast to a first-price winner-pay auction in this environment. Funk(1996) shows that there exists a pure-strategy equilibrium in the environment described above, in which player 2 wins the prize with a bid of $r_{2j}, j \in \{1,3\}$.

Proposition 1 (Moderation does not drive out extremism)**.** *In any equilibrium of* Γ_{21}*, both radicals actively participate in the conflict.*

Proof. By way of contradiction, assume that one of the radical players stays out of the conflict; without loss of generality let that player be player 1, i.e. $F_1(0) = 1$. Given player 1's strategy players 2 and 3 would randomize up to $r_{23} = r_{32} < r_{31} = r_{13}$. Player 1's payoff if he bids zero will be in the interval (v_{13}, v_{12}) and he could strictly improve upon this by bidding r_{23} which would guarantee him a payoff of $v_{11} - r_{23} = v_{11} - r_{12} = v_{12}$. □

Proposition 2 (Extremism drives out moderation)**.** *There exists an unique equilibrium of Γ_{21}. In the equilibrium of Γ_{21} the centrist stays out and the radical players randomize continuously up to their common reach, $r_{13} = r_{31}$. The two radical players apply identical strategies in equilibrium.*

Proof of Proposition 2 is provided in Appendix A.1.

One implication of Propsition 2 is that an equilibrium in which only two radicals actively participate and all centrists stay out of the conflict exists even if the population share of the radical players is much smaller than that of the centrist players. This observation is formally stated in Proposition 3.

Proposition 3 (Extremism drives out moderation with many centrists)**.** *Suppose Γ_{21} is altered by adding more players who are centrists, while maintaining the identical radical positions of players 1 and 3. Then the equilibrium described in Proposition 2 part (i) persists: all centrists stay out and the radical players 1 and 3 actively participate by randomizing continuously up to the common reach $r_{13} = r_{31}$.*

Proof. Let player m be an additional player, who is centrist in comparison with players 1 and 3. Then $r_{mj} \leq r_{jk}$ for all $j, k \in \{1, 3\}, j \neq k$, and there exists a $j \in \{1, 3\}$ such that the inequality is strict. Note that in order for players 1 and 3 to remain radical, $r_{ml} < r_{13}$ $\forall m, l \in I \backslash \{1, 3\}$. If player m bids zero and all other players follow the equilibrium strategies described in Proposition 3, then m's expected payoff is $\frac{1}{2}(v_{m1} + v_{m3})$. If player m places a strictly positive bid, $b \leq r_{jk}$, while all other players follow the strategies described in

Proposition 3, m's expected payoff would be

$$\begin{aligned}
u_m(b) &= F(b)^2 v_{mm} + (1 - F(b)^2)\left[\frac{v_{m1} + v_{m3}}{2}\right] - b \\
&= \frac{v_{m1} + v_{m3}}{2} + F(b)^2\left[\frac{r_{m1} + r_{m3}}{2}\right] - b \\
&= \frac{v_{m1} + v_{m3}}{2} + \left(\frac{b}{r_{jk}}\right)^2\left[\frac{r_{m1} + r_{m3}}{2}\right] - b \\
&= \frac{v_{m1} + v_{m3}}{2} + b\underbrace{\left[\underbrace{\frac{b}{r_{jk}}}_{\leq 1}\underbrace{\left(\frac{\frac{1}{2}(r_{m1} + r_{m3})}{r_{jk}}\right)}_{<1} - 1\right]}_{<0} \\
&< \frac{v_{m1} + v_{m3}}{2}.
\end{aligned}$$

If player m bids more than r_{jk}, then his payoff is

$$u_m(b) = v_{mm} - b < v_{mm} - r_{jk} < v_{mm} - \frac{1}{2}(r_{m1} + r_{m3}) = \frac{v_{m1} + v_{m3}}{2}.$$

Consequently, player m optimally bids zero and all centrists stay out. □

Before proceeding to the next case, we provide an example which illustrates the results above and allows us to compare the all-pay auction with a different form of all-pay contest, namely a Tullock-type model with a lottery contest success function. For the purpose of comparison we consider an example given by Linster(1993) which applies to this setting.

Example 1. Consider three players and normalize the value of the prize to one. Players' valuations are $v_1 = (1, \gamma, \gamma^2), v_2 = (\gamma, 1, \gamma), v_3 = (\gamma^2, \gamma, 1)$ where $\gamma \in [0, 1)$. The order of players' reaches is illustrated in the following diagram (Figure 2), which shows that player 2 is a centrist player and players 1 and 3 are radical. By Proposition 2 in the unique equilibrium player 2 stays out of the conflict, i.e. $F_2(x) = 1$ for all $x \geq 0$, and players 1 and 3 randomize

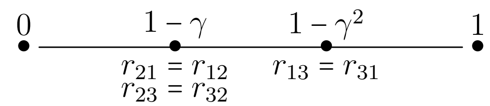

Figure 2: Illustration of players' preferences in Example 1.

symmetrically over $[0, r_{13}]$ using the cdf

$$F_1(x) = F_3(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{1-\gamma^2} & 0 \leq x \leq 1-\gamma^2 \\ 1 & x > 1-\gamma^2 \end{cases} .$$

In this environment the outcome that the centrist, player 2, wins is socially optimal in the sense that it maximizes the sum of all players' valuations[8]. In the unique equilibrium of the all-pay auction described above, this socially optimal outcome will be achieved with probability zero as compared to a probability equal to $(1-\gamma)/(3-\gamma)$ in the Tullock game with lottery contest success function as considered by Linster(1993). Moreover, the expected sum of bids is strictly greater in the all-pay auction $\left(1-\gamma^2 > \frac{2}{3-\gamma} \cdot (1-\gamma)\right)$, although the centrist submits a strictly positive bid $\left(2\frac{(1-\gamma)^2}{(3-\gamma)^2} > 0\right)$ in the lottery contest.

2.2 One Radical

Now consider a three player setup with only one radical player. Without loss of generality assume player 2 is the radical player and that the two centrist players, 1 and 3, are symmetric.[9] We refer to this game as Γ_{12}. Figure 3 illustrates players' preferences over outcomes.

[8]Generally the concept of social welfare additionally takes expenditure into account. We follow Jehiel and Moldovanu (2006) and Linster (1993) by using the sum of valuations to measure social welfare in a context of contests with identity-dependent exernalities. This interpretation implies that the players' expenditures are considered transfers. In some conflicts which are covered by our model, e.g. political lobbying, expenditures are often more accurately viewed as a social waste of resources. Therefore, we additionally discuss the effects of the auction CSF on expected total expenditure. We further elaborate on this issue in the conclusion.

[9]If players 1 and 3 were not symmetric, the identity of the player who stays out in the equilibrium described in Proposition 5 would be uniquely determined.

We find that there exists an equilibrium in Γ_{12} in which one of the centrist players stays out

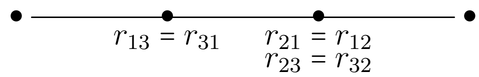

Figure 3: The case of one radical player and two centrists.

of the contest, while the radical player always actively participates in equilibrium. Moreover, even in a symmetric equilibrium (in which all players participate) extremism persists.

Proposition 4 (Moderation does not drive out extremism)**.** *In* Γ_{12}*, the radical always actively participates in the conflict.*

Proof. Assume that player 2 stays out of the contest. Then his expected payoff would be $v_{2j}, j \in \{1,3\}$, and players 1 and 3 would randomize uniformly over $[0, r_{jk}]$, $j, k \in \{1,3\}, j \neq k$. Thus, if player 2 would bid $x = r_{jk}$, he would win with certainty and receive expected payoff $v_{22} - r_{jk} > v_{22} - r_{2j} = v_{2j}$. □

Proposition 5 (Extremism drives out moderation)**.** *In* Γ_{12}*, there exists an equilibrium in which one of the centrist players stays out of the conflict, i.e.* $\exists i \in \{1,3\} : F_i(x) = 1$ *for all* $x \geq 0$*.*

Proof. Without loss of generality assume that $F_1(x) = 1$ for all $x \geq 0$. From the standard arguments for all-pay auctions (Baye et al., 1996) players 2 and 3 randomize uniformly over $[0, r_{23}]$. Both players must earn their equilibrium payoff at the upper bound of the support of their equilibrium strategies. Thus, player 3's expected payoff from a bid $x \in [0, r_{23}]$ must be v_{32} and player 2's expected payoff from a bid $x \in [0, r_{23}]$ must be v_{23}. Consequently, players 2's and 3's equilibrium strategies are

$$F_2(x) = F_3(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{r_{23}} & 0 \le x \le r_{23} \\ 1 & x > r_{23} \end{cases}.$$

Given these strategies it is optimal for player 1 to bid zero and receive expected payoff $\frac{1}{2}(v_{12} + v_{13})$, because any bid $x \in (0, r_{13})$ would yield an expected payoff of

$$\begin{aligned} u_1(x, F_{-1}) = & -x + v_{11}F_2(x)F_3(x) + v_{13}F_2(x)(1 - F_3(x)) + v_{12}F_3(x)(1 - F_2(x)) \\ & + v_{13}\int_x^{r_{13}}(1 - F_3(s))f_2(s)ds + v_{12}\int_x^{r_{13}}(1 - F_2(s))f_3(s)ds \\ = & \frac{v_{12} + v_{13}}{2} - x\left[1 - \frac{x}{r_{31}^2}\left(v_{11} - \frac{v_{12} + v_{13}}{2}\right)\right] < \frac{v_{12} + v_{13}}{2}. \end{aligned}$$

□

In the environment with one radical player, it is not necessary that a centrist player completely stays out of the contest. However, even in the symmetric equilibrium in which both centrists actively participate, extremism is present.

Proposition 6 (Extremism)**.** *In Γ_{12}, there exists a symmetric equilibrium (in the sense that identical players use identical strategies). This equilibrium exhibits extremism, and the radical player expends more effort than any centrist player in the sense of first order stochastic dominance.*

Proof of Proposition 6 is provided in Appendix A.2.

Linster's (1993) second example takes on exactly this configuration of preferences assuming the lottery CSF. In Example 2 we compare his results to those obtained when applying the auction CSF.

Example 2. Consider three players and normalize the value of the prize to one. Players' valuations are $v_1 = (1, 0, \gamma), v_2 = (0, 1, 0)$, and $v_3 = (\gamma, 0, 1)$, where $\gamma \in [0, 1)$. The order of players' reaches is illustrated in Figure 4, showing that player 2 is a radical player and players 1 and 3 are centrists. For the lottery CSF, Linster(1993) computes for this example that

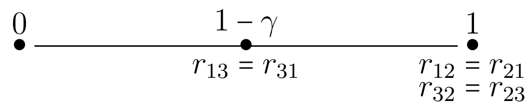

Figure 4: Illustration of players' preferences in Example 2.

the centrists bid $\frac{2}{(3+\gamma)^2}$ each and the radical bids $(1 + \gamma)\frac{2}{(3+\gamma)^2}$. The expected sum of bids is $2/(3 + \gamma) \in \left(\frac{1}{2}, \frac{2}{3}\right]$, and player 2 wins with probability $\frac{1+\gamma}{3+\gamma} \in \left[\frac{1}{3}, \frac{1}{2}\right)$, which is increasing in γ.

On the other hand, in the asymmetric equilibrium of the all-pay auction (described in Proposition 5) players expend on average higher effort (the expected sum of bids is 1) and the least socially desirable outcome, i.e. player 2 wins, is more likely to occur (2 wins with probability $\frac{1}{2}$). The symmetric equilibrium yields higher payoffs to the players who participate in the asymmetric equilibrium in which one centrist stays out. In this example all players have equal expected payoff, $\left(\frac{\gamma}{2}\right)^{\frac{2}{2-\gamma}}$, in the symmetric equilibrium, while both active players in the asymmetric equilibrium have an expected payoff of zero. The centrist who stays out receives in expectation $\frac{\gamma}{2} > \left(\frac{\gamma}{2}\right)^{\frac{2}{2-\gamma}}$ in the asymmetric equilibrium. However, the sum of expected payoffs is strictly greater in the symmetric equilibrium. In fact, the sum of expected payoffs in the symmetric equilibrium of the all-pay auction exceeds the sum of expected payoffs in the lottery contest when externalities are large enough ($\gamma \geq 0.3$).

2.3 No Radicals

Under the assumption of symmetric antagonism, there is only one three-player environment without any radical players. All reaches must coincide, $r_{ij} = r_{kl}$, $\forall i, j, k, l \in I$, $i \neq j$, $k \neq l$.

This case is illustrated in Figure 5.

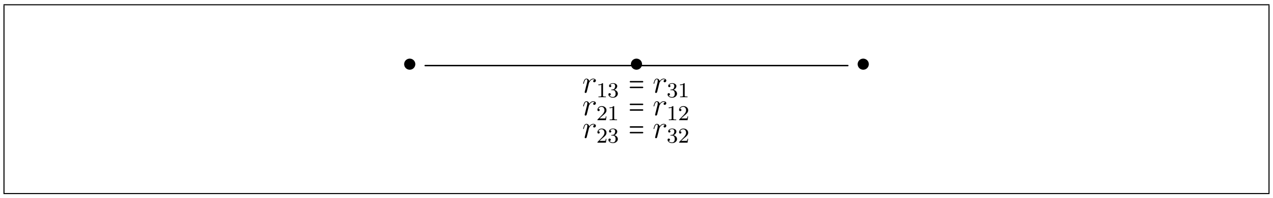

Figure 5: The case of symmetric antagonism and no radical players.

This case is equivalent to a three player all-pay auction without identity-dependent externalities in which players are symmetric and value the prize at r_{ij}, $i, j \in I$, $i \neq j$. Baye, Kovenock and DeVries (1996) show that there exists a unique symmetric equilibrium as well as a continuum of asymmetric equilibria. All equilibria however yield the same expected payoffs ($v_{ij}, i \neq j$ after rescaling) for each player and the same expected total expenditures.

3 Conclusion

In this paper, we demonstrated that the distribution of player preferences substantially influences players' behavior in all-pay auctions with identity-dependent externalities. Specifically, we showed that in these contests extremism, characterized by a higher per capita expenditure by radicals than centrists, may prevail to such an extent that radicals may expend more in the aggregate than centrists, even if they are relatively small in number. In fact, centrists may in the aggregate expend zero, even if they vastly outnumber radicals.

One consequence of this behavior is that, radical outcomes may occur with greater frequency than centrist outcomes, even in environments with a small ratio of radicals to centrists. In fact, as demonstrated in Proposition 3, centrists may vastly outnumber radicals and expend zero in the aggregate, yielding a radical outcome with certainty.

In these conflicts there is no uniform benchmark for the analysis of welfare. The discussion of welfare in the literature to date has focused on the likelihood that the final outcome maximizes the sum of the players' valuations (see for instance Linster, 1993, and Jehiel and Moldovanu, 2001). This measure appears to extend in a natural way the definition of efficiency from the auction literature (e.g. Maskin, 2000) to contests with identity-dependent externalities. Since a moderate outcome in our examples maximizes the sum of the valuations among all potential outcomes (i.e., individual player positions), social optimality is unlikely to result. Konrad (2006) points out that the sum of the players' valuations is an appropriate measure of social welfare in cases in which effort is simply a transfer. Of course, the literature on rent seeking following Tullock (1967) has viewed at least part of the expenditure in a conflict to be social waste (see for instance Tullock, 1980, and Fudenberg and Tirole, 1987). In this case the existence of radicals, by tending to increase expenditure, also increases whatever waste might arise from those expenditures.

We presented two examples that illustrated similarities (e.g., existence of extremism) as well as differences (e.g., participation vs. non-participation of centrists) that resulted from employing an auction contest success function rather than the lottery contest success function, which is prominent in the literature. Our results illustrate the importance of the choice of the institutions of conflict, as modeled by the contest success function, in determining the role of extremism and moderation in economic, political and social environments.

There are several extensions of our model that follow immediately from our analysis and address specific assumptions. One assumption made throughout the paper is that individuals do not form coalitions to promote a group's position but rather expend resources to promote an individually preferred outcome. However this is, in fact, also consistent with standard models of non-cooperative behavior in coalitions. Indeed, under the common assumption that individuals within a coalition choose their strategies non-cooperatively, our analysis does in fact apply to all-pay auctions between exogenously determined groups[10] for a group-specific public-good prize[11]. Baik et al. (2001) show that if two groups compete in an all-pay auction

[10]The assumption that groups are exogenously determined is common in the literature on contests between groups. The strategic formation of groups in contests is addressed in e.g. Baik and Lee (2001), Skaperdas (1998).

[11]In this type of contest a group's probability of winning depends on their total effort and all members of

for a public-good prize, then there exists a Nash equilibrium in which only the player who has the highest valuation of the prize within his group actively participates in the conflict, and all remaining group members free ride. This result can be generalized to three and more groups and to all-pay auctions with identity-dependent externalities. In particular, in a non-cooperative conflict between three groups of individuals who have heterogeneous valuations of the outcome promoted by their group and the other two alternatives, there exists a Nash equilibrium in which (at most) the individual with the highest willingness to bid within each group will actively participate in equilibrium. Therefore, our results for the three player model also apply to conflict between three groups when players make their decisions non-cooperatively and, thus, may free ride on other group members' efforts. By assumption, Esteban and Ray (1999) disallow free riding. In their model group members' preferences are homogenous and all players within the same group choose identical effort levels. This together with their assumption of identical convex cost technologies result in cost advantages of larger groups, which may be significant enough to result in moderation.

Another assumption that we maintain throughout our analysis is that players face identical cost functions. With heterogeneous costs the notion of player i's reach with respect to player j needs to be adjusted in order to accurately reflect the player's willingness to bid. Let player i's cost of bidding be given by a continuous, strictly increasing, unbounded function $c_i : \mathbb{R}_+ \to \mathbb{R}_+$, with $c_i(0) = 0$. Then player i's reach with respect to player j is $r_{ij} = c_i^{-1}(v_{ii} - v_{ij})$.

With heterogeneous costs the reaches r_{ij} generally do not satisfy Assumption 2, so we omit a formal analysis.[12] Moreover, it is clear that such an analysis is somewhat more

the winning group receive their valuation of winning (e.g. Baik (1993), Baik et al. (2001), Esteban and Ray (1999)). Alternatively groups may compete for a private-good prize, whereupon the prize allocated within the winning group through a second stage contest (e.g. Katz and Tokatlidu (1996), Konrad and Kovenock (2009)) or a previously determined sharing rule (Baik and Lee, 2001)

[12]See, however, Klose and Kovenock (2012) for an analysis of all-pay auctions with identity-dependent externalities and more general preference structures.

complicated in the presence of identity-dependent externalities than in the original analysis of Siegel (2009, 2010). First, we do not have the generic uniqueness of equilibrium payoffs (Siegel(2009)) to aid in tying down distributions. Second, because the payoff of a player at any bid b depends not only on the probability that he is outbid at that bid, but on the conditional probability that each of the other players is the highest bidder, there is no obvious extension of the Siegel (2010) algorithm to pin down equilibrium distributions or the set of active bidders.

Nonetheless, we can say something about certain classes of asymmetric cost functions. Suppose, for instance, that players have cost functions of the form employed by Moldovanu and Sela (2001) and in Siegel's (2010) analysis of *simple contests*, $c_i(b) = \gamma_i C(b)$, where $\gamma_i > 0$ for all $i \in I$ and $C(b)$ is continuous, strictly increasing, and unbounded with $C(0) = 0$. Moreover, suppose that in the game Γ_{21} described in section 2.1, the two radicals (based on preferences over outcomes in the original game Γ_{21} with cost $C(b) = b$) have a common coefficient of cost, $\gamma_i = \gamma_R, i = 1, 3$, which is strictly less than the corresponding coefficient of the single centrist, $\gamma_2 = \gamma_N$. Then the results of Propositions 1 and 2 continue to hold, with proofs modified to account for the fact that, with the cost asymmetry, $r_{21} = r_{23} < r_{32} = r_{12}$. Similarly, the result of Proposition 3 of the game with two radicals and multiple centrists would continue to hold if any centrist player i (based on the preferences in the original game) has a coefficient $\gamma_i \geq \gamma_R$. On the other hand, if an original centrist exhibits low enough costs of effort this may cause him to actively participate in the conflict, and may improve welfare.

In a similar fashion, Propositions 4-6 in section 2.2 continue to hold under the assumption that the two centrists in the game Γ_{12} have an identical coefficient of cost $\gamma_i = \gamma_N, i = 1, 3$, which is greater than the coefficient of the sole radical, $\gamma_2 = \gamma_R$. Of course, there are other formulations of cost for which similar results arise. One example is an appropriate choice of budget constraints. In fact, a general extension to budget constrained costs along the lines of Che and Gale's (1998) analysis of standard all-pay auctions raises new and interesting phenomena. For instance, if a sufficiently small budget constraint is imposed upon a radical player, the likelihood that a centrist wins the conflict may increase, thereby increasing the expected sum of valuations from the resulting outcome. These and other explorations of all-pay auctions with identity-dependent externalities are left for future research.

References

[1] Baik, K.H., 1993. "Effort Levels in Contests: The Public-Good Prize Case". Economics Letters, Vol. 41, Issue 4, pp. 363-367.

[2] Baik, K.H., Kim, I.-G., and Na, S., 2001. "Bidding for a group-specific public-good prize".Journal of Public Economics, Vol. 82, Issue 3, pp. 415-429.

[3] Baik, K.H., and Lee, S., 2001. "Strategic Groups and Rent Dissipation". Economic Inquiry, Vol. 39, Issue 4, pp. 672-684.

[4] Baye, M., Kovenock, D., and DeVries, C., 1996. "The all-pay auction with complete information". Economic Theory, Vol. 8, pp. 291-305.

[5] Che, Y.-K., and I. Gale, 1998. "Caps on Political Lobbying". The American Economic Review, Vol. 88, No. 3, pp. 643-651.

[6] Esteban, J., and Ray, D., 1999. "Conflict and Distribution". Journal of Economic Theory, Vol. 87, pp. 379-415.

[7] Fudenberg, D. and J. Tirole, 1987, "Understanding Rent Dissipation: On the Use of Game Theory in Industrial Organization," American Economic Review 77 (Papers and Proceedings of AEA), 176-183.

[8] Funk, P., 1996. "Auctions with Interdependent Valuations". International Journal of Game Theory, Vol. 25, pp. 51-64.

[9] Janssen, M. and B. Moldovanu, 2004, "Allocation Mechanisms and Post-Allocation Interaction," in Auctioning Public Assets, M.Janssen (ed.), Cambridge: Cambridge University Press.

[10] Jehiel, P. and B. Moldovanu, 1996, "Strategic Non-participation," Rand Journal of Economics, Vol. 27, pp. 84-98.

[11] Jehiel, P. and B. Moldovanu, 2001, "Efficient Design with Interdependent Valuations," Econometrica, Vol.69, Issue 5, pp. 1237-1259.

[12] Jehiel, P.,and B. Moldovanu, 2006, "Allocative and Informational Externalities in Auctions and Related Mechanisms". in The Proceedings of the 9th World Congress of the Econometric Society, edited by Richard Blundell, Whitney Newey, and Torsten Persson, Cambridge University Press, 2006.

[13] Katz, E., and Tokatlidu, J., 1995. "Group competition for rents". European Journal of Political Economy, Vol. 12, pp. 599-607.

[14] Klose, B. and D. Kovenock, 2012, "The All-Pay Auction with Identity-Dependent Externalities". Unpublished.

[15] Konrad, K.A., 2006. "Silent Interests and All-pay Auctions". International Journal of Industrial Organization, Vol. 24, pp. 701-713.

[16] Konrad, K.A., 2009. "Strategy and Dynamics in Contests". Oxford University Press.

[17] Linster, B.G., 1993. "A Generalized Model of Rent-Seeking Behavior". Public Choice, Vol. 77, Issue 2, pp. 421-435.

[18] Maskin, E.S. (2000), "Auctions, Development, and Privatization: Efficient Auctions with Liquidity-Constrained Buyers," European Economic Review, Vol.44, Issues 4-6, pp.667-681.

[19] Moldovanu, B. and A. Sela, 2001, "The Optimal Allocation of Prizes in Contests". The American Economic Review , Vol. 91, No. 3, pp. 542-558.

[20] Osborne, M. J., Rosenthal, J.S., and Turner, M.A., 2000. "Meetings with Costly Participation". The American Economic Review, Vol. 90, No. 4, pp. 927-943.

[21] Siegel, R., 2009. "All-pay Contests". Econometrica, Vol. 77, Issue 1, pp. 71-92.

[22] Siegel, R., 2010. "Asymmetric Contests with Conditional Investments." American Economic Review, Vol. 100, No. 5, pp. 2230-2260.

[23] Skaperdas, S., 1998. "On the formation of Alliances in Conflict and Contests". Public Choice, Vol. 96, Issue 1/2, pp. 25-42.

[24] Tullock, G., 1980."Efficient Rent Seeking". In Toward a Theory of the Rent Seeking Society, J.M. Buchanan, R. Tollison and G. Tullock, eds., Texas A&M Press, 269-82.

A Appendix

A.1 Proof of Proposition 2

Proof. In a first step we show existence by constructing an equilibrium, we then show uniqueness of the equilibrium described before in a second step involving multiple lemmas.

The strategy profile in which 2 stays out completely (puts mass 1 on zero) and players 1 and 3 randomize uniformly over $[0, r_{jk}]$ ($j, k \in \{1,3\}$, $j \neq k$) is a Nash equilibrium. Assume that 2 uses the strategy $F_2(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x \geq 0 \end{cases}$. Then (by Baye et al., 1996) it is optimal for players 1 and 3 to randomize over $[0, r_{jk}]$ according to

$$F(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \frac{x}{r_{jk}} & \text{for} \quad 0 \leq x \leq r_{jk} \\ 1 & \text{for} \quad x > r_{jk} \end{cases}.$$

Given that 1 and 3 apply this strategy player 2's payoff if he submits a strictly positive bid $x \in (0, r_{2j}]$ is:

$$\begin{aligned} u_2^*(x) &= [F(x)]^2 v_{22} + (1 - [F(x)]^2) v_{2j} - x \\ &= v_{2j} + F(x)^2 r_{2j} - x \\ &= v_{2j} - x\left(1 - \frac{x}{r_{jk}}\frac{r_{2j}}{r_{jk}}\right) \\ &< v_{2j}. \end{aligned}$$

It is therefore a best response for player 2 to stay out of the conflict.

Next we prove the uniqueness of the equilibrium described in Proposition 2, by first showing that any equilibrium of Γ_{21} is symmetric in the sense that both radicals (players 1 and 3) choose identical strategies. In a second step we then show that the set of symmetric equilibria of Γ_{21} is a singleton, given by the cut-throat competition equilibrium described above.

Let $\underline{s}_i$ and $\bar{s}_i$ be the lower and upper bound, respectively, of the support of an equilibrium strategy for player i, $i \in I$, and define $\bar{s} = \max_{i \in I}\{\bar{s}_i\}$. In the following, the indices j and k refer to two different radical players, i.e. $j \in \{1,3\}, k \in \{1,3\} \backslash \{j\}$.

Lemma A.1. *$\underline{s}_i = 0$ for all $i \in I$, and for at least one player $l \in I$, $F_l(0) = 0$.*

Proof. Assume $\underline{s}_i > \underline{s}_l \geq \underline{s}_m \geq 0$ for some $i, l, m \in I$. Any bid $x \in [0, \underline{s}_i)$ results in a loss with certainty. Therefore, players l and m do not put mass anywhere over $(0, \underline{s}_i)$. Moreover, no player l or m can place a mass point at $\underline{s}_i$, because if two or more players had a mass point at $\underline{s}_i$, then one could improve by moving mass up, and if only one player had a mass point at $\underline{s}_i$, then he would improve by moving the mass down. Altogether players l and m do not put mass anywhere over $(0, \underline{s}_i]$, but then player i would improve by moving mass down. This contradiction implies that there exist mutually different $i, l, m \in I$ such that $\underline{s}_i = \underline{s}_l \geq \underline{s}_m \geq 0$. Assume that $\underline{s}_i = \underline{s}_l > \underline{s}_m \geq 0$ for some $i, l, m \in I$. It cannot be the case that both players, i and l, have a mass point at $\underline{s}_i = \underline{s}_l$ (otherwise one could improve by moving mass up slightly), but then at least one of them would win with probability arbitrarily close to zero in some neighborhood above $\underline{s}_i$ and would be better off by moving mass down to zero. It follows that in equilibrium $\underline{s}_1 = \underline{s}_2 = \underline{s}_3 = \underline{s}$. It cannot be the case that all three players have a mass point at $\underline{s}$ otherwise a player could improve by moving this mass up slightly. Therefore, at least one player loses with certainty at $\underline{s}$. Altogether this shows that $\underline{s}_1 = \underline{s}_2 = \underline{s}_3 = 0$. □

Lemma A.2. *There are no mass points at x in any player's equilibrium distribution $\forall x \in (0, \bar{s}]$.*

Proof. Suppose player $i \in I$ has a mass point at $x \in (0, \bar{s}]$. Since, from Lemma A.1, $F_l(x) > 0$ for every $l \in I$, for sufficiently small $\epsilon > 0$ no player $j \neq i$ would place mass in $(x - \epsilon, x]$ since that player could improve his payoff by moving mass from that interval to infinitesimally above x. But then it is not optimal for i to put mass at x. $\square$

Lemma A.3. $\bar{s}_1 = \bar{s}_3 > \bar{s}_2$.

Proof. Obviously, it cannot be the case that $\bar{s}_i > \bar{s}_l \geq \bar{s}_m$ for some $i, l, m \in I$, because player i would strictly improve his payoff by moving mass from $\left(\frac{1}{2}(\bar{s}_l + \bar{s}_i), \bar{s}_i\right]$ down to $\frac{1}{2}(\bar{s}_l + \bar{s}_i)$. Suppose, $\bar{s}_1 = \bar{s}_2 = \bar{s}_3 = \bar{s} > 0$. Since any bid $b_2 > r_{2j}$ of player 2 is strictly dominated by $b_2 = 0$ it follows that $\bar{s} \leq r_{2j}$. By Lemma A.1 $\underline{s}_i = 0$ for all $i \in I$ and at most two players may have a mass point at zero. Therefore, there exists a radical player j, who is outbid with certainty when bidding zero and whose payoff from bidding zero is $u_j^*(0) = \alpha v_{j2} + (1-\alpha) v_{jk}$ for some $\alpha \in (0,1)$. By assumption $r_{jk} > r_{j2}$ which implies by definition that $v_{jk} < v_{j2}$. Then, (by Lemma A.1) player j's expected equilibrium payoff would be $u_j^* < v_{j2}$. On the other hand, by submitting a bid $\bar{s} + \epsilon$ greater than $\bar{s}$ player j would receive $u_j^*(\bar{s} + \epsilon) = v_{jj} - \bar{s} - \epsilon \geq v_{jj} - r_{j2} - \epsilon = v_{j2} - \epsilon$. Therefore, by choosing $\epsilon > 0$ small enough, he would improve his payoff. Thus, $\bar{s}_1 = \bar{s}_2 = \bar{s}_3$ cannot hold true. By the same argument it cannot be the case that $\bar{s}_j < \bar{s}_2 = \bar{s}_k = \bar{s}$. Hence, $\bar{s}_j = \bar{s}_k > \bar{s}_2$. $\square$

Lemma A.4. $\bar{s}_2 < r_{2j}, j \in \{1, 3\}$.

Proof. By Lemma A.3 player 2 loses with strictly positive probability at $\bar{s}_2$. Suppose $\bar{s}_2 \geq r_{2j}$, then player 2's equilibrium payoff at $\bar{s}_2$ is

$$\begin{aligned} u_2^*(\bar{s}_2, F_j, F_k) &= [F_j(\bar{s}_2) \cdot F_k(\bar{s}_2)] v_{22} + (1 - [F_j(\bar{s}_2) \cdot F_k(\bar{s}_2)])\, v_{2j} - \bar{s}_2 \\ &\leq v_{2j} - \underbrace{(1 - [F_j(\bar{s}_2) \cdot F_k(\bar{s}_2)])}_{>0 \text{ by Lemma A.3}} r_{2j} < v_{2j}. \end{aligned}$$

This is a contradiction, because player 2 could guarantee himself a payoff of at least v_{2j} by bidding zero. $\square$

Lemma A.5. *Players $j \in \{1,3\}$ earn expected equilibrium payoffs $v_{jj} - \bar{s}$.*

Proof. From Lemmas A.2 and A.3 players 1 and 3 must earn their expected equilibrium payoff at the upper bound of the support of their mixed strategies, $\bar{s}$, and neither has a mass point at $\bar{s}$. Therefore, their expected equilibrium payoff is $u_j^* = v_{jj} - \bar{s}$. □

Lemma A.6. $F_1(x) = F_3(x)$ *for all* $x \in [\bar{s}_2, \bar{s}]$.

Proof. Notice that $F_2(x) = 1$ for all $x \in [\bar{s}_2, \bar{s}]$, and $F_1(\bar{s}) = F_3(\bar{s}) = 1$. From Lemma A.3, for $x \in (\bar{s}_2, \bar{s}]$

$$u_j(x, F_2, F_k) = F_k(x)v_{jj} + (1 - F_k(x))v_{jk} - x = v_{jk} + F_k(x)r_{jk} - x.$$

By Lemma A.5 it follows that

$$\begin{aligned} v_{jk} + F_k(x)r_{jk} - x &= v_{jj} - \bar{s} \\ \Leftrightarrow F_k(x) &= 1 - \frac{\bar{s} - x}{r_{jk}} \end{aligned}$$

and by Assumption 2 (symmetric inter-agent antagonism) follows that players j and k use identical strategies $F_j(x) = F_k(x) = 1 - \frac{\bar{s}-x}{r_{jk}}$ over the interval $(\bar{s}_2, \bar{s}]$. If $\bar{s}_2 > 0$, then by Lemma A.2 this holds over $[\bar{s}_2, \bar{s}]$. If $\bar{s}_2 = 0$ right-continuity of $F_i, i \in I$, implies $F_1(0) = F_3(0)$. □

Lemma A.7. *For any nondegenerate interval $[\underline{t}, \bar{t}] \in [0, \bar{s}]$ ($\underline{t} < \bar{t}$) there are at least two players, $i, j \in I$, such that $F_l(\bar{t}) - F_l(\underline{t}) > 0$ for $l = i, j$.*

Proof. Suppose there is a $t > \underline{t}$ such that $F_i(t) - F_i(\underline{t}) = 0$ for all $i \in I$, and let $\bar{t}$ be the supremum over all t with this property, i.e. define $\bar{t} = \sup\{t > \underline{t} : F_i(t) - F_i(\underline{t}) = 0 \text{ for all } i \in l\}$. Notice that by Lemma A.1 $\underline{t} > 0$. Since $\bar{t} > \underline{t} \geq 0$ no player has a mass point at $\bar{t}$ by Lemma A.2. Let player $i \in I$ and $m, l \in I\backslash\{i\}$, then player i's payoff from a bid $\bar{t} + \epsilon$ is

$$u_i(\bar{t} + \epsilon, F_l, F_m) = v_{ii} \cdot F_l(\bar{t} + \epsilon)F_m(\bar{t} + \epsilon) + v_{il} \int_{\bar{t}+\epsilon}^{\bar{s}} F_m(y) f_l(y) dy + v_{im} \int_{\bar{t}+\epsilon}^{\bar{s}} F_l(y) f_m(y) dy - \bar{t} - \epsilon.$$

On the other hand player i's payoff from bidding $\underline{t}$ is

$$\begin{aligned} u_i(\underline{t}, F_l, F_m) &= v_{ii} \cdot F_l(\underline{t}) F_m(\underline{t}) + v_{il} \int_{\underline{t}}^{\bar{s}} F_m(y) f_l(y) dy + v_{im} \int_{\underline{t}}^{\bar{s}} F_l(y) f_m(y) dy - \underline{t} \\ &= v_{ii} \cdot F_l(\bar{t}) F_m(\bar{t}) + v_{il} \int_{\bar{t}}^{\bar{s}} F_m(y) f_l(y) dy + v_{im} \int_{\bar{t}}^{\bar{s}} F_l(y) f_m(y) dy - \underline{t}, \end{aligned}$$

which is strictly greater than $u_i(\bar{t} + \epsilon, F_l, F_m)$ for $\epsilon > 0$ sufficiently small. Thus, for small enough $\epsilon > 0$ a player would improve his payoff by moving mass from $[\bar{t}, \bar{t} + \epsilon]$ to $\underline{t}$. Therefore, no $t > \underline{t}$ such that $F_i(t) - F_i(\underline{t}) = 0$ for all $i \in I$ exists.

Suppose that there is only one player $i \in I$ with $F_i(\bar{t}) - F_i(\underline{t}) > 0$, and denote the other two players by $l, m \in I \backslash \{i\}$. Note that for players $p \in \{l, m\}$, $f_p(t) = 0$ for all $t \in (\underline{t}, \bar{t})$ and $F_p(\underline{t}) = F_p(t) = F_p(\bar{t})$ for all $t \in (\underline{t}, \bar{t})$. Player i's expected payoff from a bid $t \in (\underline{t}, \bar{t})$ is

$$\begin{aligned} u_i(t, F_l, F_m) &= v_{ii} \cdot F_l(t) F_m(t) + v_{il} \int_{t}^{\bar{s}} F_m(y) f_l(y) dy + v_{im} \int_{t}^{\bar{s}} F_l(y) f_m(y) dy - t \\ &= v_{ii} \cdot F_l(\underline{t}) F_m(\underline{t}) + v_{il} \int_{\underline{t}}^{\bar{s}} F_m(y) f_l(y) dy + v_{im} \int_{\underline{t}}^{\bar{s}} F_l(y) f_m(y) dy - t \\ &< v_{ii} \cdot F_l(\underline{t}) F_m(\underline{t}) + v_{il} \int_{\underline{t}}^{\bar{s}} F_m(y) f_l(y) dy + v_{im} \int_{\underline{t}}^{\bar{s}} F_l(y) f_m(y) dy - \underline{t} \\ &= u_i(\underline{t}, F_l, F_m). \end{aligned}$$

Therefore, player i could improve his payoff by moving mass from the interval $(\underline{t}, \bar{t}]$ to its lower bound $\underline{t}$. □

Lemma A.8. $F_1(x) = F_3(x)$ *for all* $x \in [0, \bar{s}]$.

Proof. If $\bar{s}_2 = 0$ then $F_1(x) = F_3(x)$ for all $x \in [0, \bar{s}]$ by A.6, thus we assume in the following that $\bar{s}_2 > 0$. For any bid $b_j > 0$ in the support of player j's equilibrium strategy his expected payoff must be equal to $v_{jj} - \bar{s}$ (by Lemma A.5). That is:[13]

$$\begin{aligned}
v_{jj} - \bar{s} &= v_{jj} \cdot (1 - p\{2 \text{ wins} \mid b_j\} - p\{\text{k wins} \mid b_j\}) + v_{j2} \cdot p\{2 \text{ wins} \mid b_j\} + v_{jk} \cdot p\{\text{k wins} \mid b_j\} - b_j \\
&= v_{jj} - r_{j2} \cdot p\{2 \text{ wins} \mid b_j\} - r_{jk} \cdot p\{\text{k wins} \mid b_j\} - b_j \\
&= v_{jj} - r_{j2} \cdot \int_{b_j}^{\bar{s}} F_k(y) f_2(y) dy - r_{jk} \cdot \int_{b_j}^{\bar{s}} F_2(y) f_k(y) dy - b_j \\
&= v_{jj} - r_{j2} \cdot \int_{b_j}^{\bar{s}} F_k(y) f_2(y) dy - r_{jk} \cdot \left([F_2(y) F_k(y)]_{b_j}^{\bar{s}} - \int_{b_j}^{\bar{s}} F_k(y) f_2(y) dy \right) - b_j \\
&= v_{jj} - (r_{j2} - r_{jk}) \cdot \int_{b_j}^{\bar{s}} F_k(y) f_2(y) dy - r_{jk} \cdot (1 - F_2(b_j) F_k(b_j)) - b_j \\
\Leftrightarrow \bar{s} - b_j &= (r_{j2} - r_{jk}) \cdot \int_{b_j}^{\bar{s}} F_k(y) f_2(y) dy + r_{jk} \cdot (1 - F_2(b_j) F_k(b_j))
\end{aligned}$$

Define α, β, γ such that $\alpha \equiv r_{12} = r_{21} = r_{32} = r_{23}$, $\beta \equiv r_{13} = r_{31}$, and $\gamma = \alpha - \beta$. Note that $\alpha, \beta > 0$ and $\gamma < 0$. Then for any $b_j, b_k \in (0, \bar{s}]$:

$$\bar{s} - b_j \leq \gamma \cdot \int_{b_j}^{\bar{s}} F_k(s) f_2(s) ds + \beta \cdot (1 - F_2(b_j) F_k(b_j)), \tag{A.1.1}$$

and

$$\bar{s} - b_k \leq \gamma \cdot \int_{b_k}^{\bar{s}} F_j(s) f_2(s) ds + \beta \cdot (1 - F_2(b_k) F_j(b_k)).,\tag{A.1.2}$$

where equality must hold in A.1.1 for bids b_j in the support of player j's equilibrium strategy and in A.1.2 for b_k in the support of player k's equilibrium strategy.

By way of contradiction, assume that there exists some $b_0 > 0$ such that $F_1(b_0) \neq F_3(b_0)$. By Lemma A.2 F_1 and F_3 are continuous everywhere on $(0, \bar{s}]$ and by Lemma A.6 $F_1(\bar{s}_2) = F_3(\bar{s}_2)$. This implies that either there exists an interval $[x, y] \subset (0, \bar{s}_2]$ such that $F_1(x) = F_3(x)$, $F_1(y) = F_3(y)$, and $F_1(b) \neq F_3(b) \forall b \in (x, y)$, or there exists $\bar{x} > b_0$ such that $F_1(b) = F_3(b)\ \forall b \geq \bar{x}$ and $F_1(b) \neq F_3(b) \forall b \in [0, \bar{x})$.

[13]We let $p\{\text{i wins} \mid b_j\}$ denote the probability that player i wins conditional on the event that player j bids b_j.

Suppose that $[x, y]$ is an interval such that $F_1(x) = F_3(x)$, $F_1(y) = F_3(y)$, and $F_1(b) \neq F_3(b) \forall b \in (x, y)$. We treat the following four cases separately:

1. $x, y \in supp_j \cap supp_k$, where $supp_i$ denotes the support of player i's equilibrium strategy. Without loss of generality let $F_j(b) > F_k(b)$ for all $b \in (x, y)$. In this case by (A.1.1) and (A.1.2) at $b = y$

$$\begin{aligned} \gamma \cdot \int_y^{\bar{s}} F_k(s) f_2(s) ds + \beta \cdot (1 - F_2(y) F_k(y)) &= \gamma \cdot \int_y^{\bar{s}} F_j(s) f_2(s) ds + \beta \cdot (1 - F_2(y) F_j(y)) \\ \Leftrightarrow \gamma \cdot \int_y^{\bar{s}} (F_j(s) - F_k(s)) f_2(s) ds &= \beta F_2(y) \cdot \underbrace{(F_j(y) - F_k(y))}_{=0} = 0 \end{aligned}$$

By definition $\gamma < 0$, hence

$$\int_y^{\bar{s}} (F_j(s) - F_k(s)) f_2(s) ds = 0.$$

Similarly, at $b = x$

$$\int_x^{\bar{s}} (F_j(s) - F_k(s)) f_2(s) ds = 0.$$

Then, by $\int_x^{\bar{s}} (F_j(s) - F_k(s)) f_2(s) ds = \int_x^y (F_j(s) - F_k(s)) f_2(s) ds + \int_y^{\bar{s}} (F_j(s) - F_k(s)) f_2(s) ds$ follows that

$$\int_x^y (F_j(s) - F_k(s)) f_2(s) ds = 0.$$

If $f_2(s) > 0$ for any $s \in (x, y)$ this contradicts $F_j(s) > F_k(s) \forall s \in (x, y)$.

If $f_2(s) = 0$ for all $s \in (x, y)$, then by Lemma A.7 $[x, y] \in supp_j \cap supp_k$ and $F_2(x) = F_2(y)$. In this case (A.1.1) and (A.1.2) simplify to

$$\begin{aligned} \bar{s} - b_j &= \gamma \cdot \int_y^{\bar{s}} F_k(s) f_2(s) ds + \beta \cdot (1 - F_2(y) F_k(b_j)), \text{ and} \\ \bar{s} - b_k &= \gamma \cdot \int_y^{\bar{s}} F_j(s) f_2(s) ds + \beta \cdot (1 - F_2(y) F_j(b_k)) \end{aligned}$$

respectively for all $b_j, b_k \in (x, y)$. Notice that in both expressions the integral is constant in the player's own bid. Since F_j and F_k coincide at x and y, $\int_y^{\bar{s}} F_k(s) f_2(s) ds = \int_y^{\bar{s}} F_j(s) f_2(s) ds$. This shows that $F_j(b) = F_k(b)\ \forall b \in (x, y)$, which contradicts our assumption.

2. $y \in supp_j \cap supp_k, x \in supp_j \backslash supp_k$.

   Then by (A.1.1) and (A.1.2) at $b = x$

$$\begin{aligned} \gamma \cdot \int_x^{\bar{s}} F_k(s) f_2(s) ds + \beta \cdot (1 - F_2(x) F_k(x)) &\leq \gamma \cdot \int_x^{\bar{s}} F_j(s) f_2(s) ds + \beta \cdot (1 - F_2(x) F_j(x)) \\ \Leftrightarrow \gamma \cdot \int_x^{\bar{s}} (F_k(s) - F_j(s)) f_2(s) ds &\leq \beta F_2(x) \cdot \underbrace{(F_k(x) - F_j(x))}_{=0} = 0 \end{aligned}$$

   By definition $\gamma < 0$, hence

$$\int_x^{\bar{s}} (F_k(s) - F_j(s)) f_2(s) ds \geq 0.$$

   If $f_2(s) > 0$ for any $s \in (x, y)$ this implies $F_j(s) < F_k(s)$, because by assumption $F_j(b) \neq F_k(b)\ \forall b \in (x, y)$ and by Lemma A.2 (no mass points) F_j and F_k are continuous. By assumption $x \notin supp_k$. So there exists an $\epsilon > 0$ such that $F_k(x + \delta) = F_k(x)$ for all δ such that $0 < \delta < \epsilon$. But then

$$F_j(x + \delta) < F_k(x + \delta) = F_k(x) = F_j(x),$$

   which is a contradiction, because F_j is a cumulative distribution function and as such is non-decreasing.

   If $f_2(s) = 0\ \forall s \in (x, y)$, then from Lemma A.7 $[x, y] \subseteq supp_j \cap supp_k$, a contradiction to the assumption $x \notin supp_k$.

3. $x \in supp_j \cap supp_k, y \in supp_j \backslash supp_k$.

By (A.1.1) and (A.1.2) at $b = y$

$$\begin{aligned} \gamma \cdot \int_y^{\bar{s}} F_k(s) f_2(s) ds + \beta \cdot (1 - F_2(y) F_k(y)) &\leq \gamma \cdot \int_y^{\bar{s}} F_j(s) f_2(s) ds + \beta \cdot (1 - F_2(y) F_j(y)) \\ \Leftrightarrow \gamma \cdot \int_y^{\bar{s}} (F_k(s) - F_j(s)) f_2(s) ds &\leq \beta F_2(y) \cdot \underbrace{(F_k(y) - F_j(y))}_{=0} = 0 \end{aligned}$$

By definition $\gamma < 0$, hence

$$\int_y^{\bar{s}} (F_k(s) - F_j(s)) f_2(s) ds \geq 0.$$

By (A.1.1) and (A.1.2) at $b = x$

$$\begin{aligned} \gamma \cdot \int_x^{\bar{s}} F_k(s) f_2(s) ds + \beta \cdot (1 - F_2(x) F_k(x)) &= \gamma \cdot \int_x^{\bar{s}} F_j(s) f_2(s) ds + \beta \cdot (1 - F_2(x) F_j(x)) \\ \Leftrightarrow \gamma \cdot \int_x^{\bar{s}} (F_j(s) - F_k(s)) f_2(s) ds &= \beta F_2(x) \cdot \underbrace{(F_j(x) - F_k(x))}_{=0} = 0. \end{aligned}$$

$\gamma < 0$, hence

$$\begin{aligned} \int_x^{y} (F_j(s) - F_k(s)) f_2(s) ds + \underbrace{\int_y^{\bar{s}} (F_j(s) - F_k(s)) f_2(s) ds}_{\leq 0} &= 0 \\ \Rightarrow \int_x^{y} (F_j(s) - F_k(s)) f_2(s) ds &\geq 0 \end{aligned}$$

If $f_2(s) > 0$ for any $s \in (x, y)$, then this implies $F_j(s) \geq F_k(s)$. By assumption, $F_j(b) \neq F_k(b) \, \forall b \in (x, y)$, thus $F_j(b) > F_k(b) \, \forall b \in (x, y)$. By assumption $y \notin supp_k$. Hence, there exists an $\epsilon > 0$ such that $F_k(y - \delta) = F_k(y) \, \forall 0 < \delta < \epsilon$. But then

$$F_j(y - \delta) > F_k(y - \delta) = F_k(y) = F_j(y),$$

a contradiction to the fact that F_j is a cumulative distribution function and as such is non-decreasing.

If $f_2(s) = 0 \, \forall s \in (x, y)$, then from Lemma A.7 $[x, y] \subseteq supp_j \cap supp_k$, a contradiction to the assumption $y \notin supp_k$.

4. $x \in supp_j \setminus supp_k$, $y \in supp_k \setminus supp_j$.

   By (A.1.1) and (A.1.2) at $b = x$

$$\begin{aligned} \gamma \cdot \int_x^{\bar{s}} F_k(s) f_2(s) ds + \beta \cdot (1 - F_2(x) F_k(x)) &\leq \gamma \cdot \int_x^{\bar{s}} F_j(s) f_2(s) ds + \beta \cdot (1 - F_2(x) F_j(x)) \\ \Leftrightarrow \gamma \cdot \int_x^{\bar{s}} (F_k(s) - F_j(s)) f_2(s) ds &\leq \beta F_2(x) \cdot \underbrace{(F_k(x) - F_j(x))}_{=0} = 0 \end{aligned}$$

   By definition $\gamma < 0$, hence

$$\int_x^{\bar{s}} (F_k(s) - F_j(s)) f_2(s) ds \geq 0.$$

   A similar argument shows that at $b = y$

$$\int_y^{\bar{s}} (F_k(s) - F_j(s)) f_2(s) ds \leq 0.$$

   Consequently,

$$0 \leq \int_x^{\bar{s}} (F_k(s) - F_j(s)) f_2(s) ds = \int_x^{y} (F_k(s) - F_j(s)) f_2(s) ds + \underbrace{\int_y^{\bar{s}} (F_k(s) - F_j(s)) f_2(s) ds}_{\leq 0}$$
$$\Rightarrow \int_x^{y} (F_k(s) - F_j(s)) f_2(s) ds \geq 0.$$

   If $f_2(s) > 0$ for any $s \in (x, y)$, then this implies $F_k(s) \geq F_j(s)$. By assumption $F_j(b) \neq F_k(b) \, \forall b \in (x, y)$, thus $F_k(b) > F_j(b) \, \forall b \in (x, y)$. By assumption $x \notin supp_k$. Hence, there exists an $\epsilon > 0$ such that $F_k(x+\delta) = F_k(x)\ \forall 0 < \delta < \epsilon$. But then

$$F_j(x+\delta) < F_k(x+\delta) = F_k(x) = F_j(x),$$

a contradiction to the fact that F_j is a cumulative distribution function and as such is non-decreasing.

If $f_2(s) = 0\ \forall s \in (x,y)$, then from Lemma A.7 $[x,y] \subseteq supp_j \cap supp_k$, a contradiction to the assumption $x \notin supp_k, y \notin supp_j$.

Taking these four possible cases together, there cannot exist any interval $[x,y]$ with $F_1(x) = F_3(x)$, $F_1(y) = F_3(y)$, and $F_1(b) \neq F_3(b)\ \forall b \in (x,y)$.

Assume now that there exists an $\bar{x} > b_0$ such that $F_j(b) = F_k(b)\ \forall b \geq \bar{x}$ and $F_j(b) > F_k(b), \forall b \in [0,\bar{x})$. Players 1 and 3 must earn their equilibrium payoff at (or arbitrarily close to) zero, so by (A.1.1) and (A.1.2)

$$\begin{aligned}
&\gamma \cdot \int_0^{\bar{s}} F_k(s) f_2(s) ds + \beta \cdot (1 - F_2(0)F_k(0)) = \gamma \cdot \int_0^{\bar{s}} F_j(s) f_2(s) ds + \beta \cdot (1 - F_2(0)F_j(0)) \\
\Leftrightarrow &\gamma \cdot \int_0^{\bar{s}} (F_j(s) - F_k(s)) f_2(s) ds - \beta F_2(0) [F_j(0) - F_k(0)] = 0 \\
\Leftrightarrow &\gamma \cdot \int_0^{\bar{x}} \underbrace{(F_j(s) - F_k(s))}_{>0} f_2(s) ds + \gamma \cdot \int_{\bar{x}}^{\bar{s}} \underbrace{(F_j(s) - F_k(s))}_{=0} f_2(s) ds - \beta F_2(0) \underbrace{[F_j(0) - F_k(0)]}_{>0} = 0
\end{aligned} \tag{A.1.3}$$

If $f_2(s) = 0$ for all $s \in (0,\bar{x})$, then (A.1.3) simplifies to

$$-\beta F_2(0) \underbrace{[F_j(0) - F_k(0)]}_{>0} = 0.$$

This implies $F_2(0) = 0$, which is a contradiction, because Lemma A.1 and $f_2(s) = 0$ for all $s \in (0,\bar{x})$ imply $F_2(0) > 0$.

If $f_2(s) > 0$ for some $s \in (0, \bar{x})$, then $\beta > 0$ and $\gamma < 0$ imply that

$$\underbrace{\gamma \cdot \int_0^{\bar{x}} (F_j(s) - F_k(s)) f_2(s) ds}_{<0} - \underbrace{\beta F_2(0) \left[F_j(0) - F_k(0)\right]}_{\geq 0} < 0,$$

a contradiction to (A.1.3). Consequently, there can exists no $b_0 > 0$ such that $F_1(b_0) \neq F_3(b_0)$. □

Lemma A.9. *$F \equiv F_1 = F_3$ first order stochastically dominates F_2.*

Proof. If $\bar{s}_2 = 0$, then $F_2(x) = 1 \, \forall x \geq 0$. Hence, F first order stochastically dominates F_2.

Therefore, assume in the following that $\bar{s}_2 > 0$. By way of contradiction assume that there exists some $b_0 \in [0, \bar{s}_2)$ such that $F_2(b_0) < F(b_0)$. Note that by Lemmas A.7 and A.8 $supp_j = [0, \bar{s}], j \in \{1, 3\}$. Furthermore, by Lemma A.3 $F_2(\bar{s}_2) > F(\bar{s}_2)$ and by Lemma A.2 no player's equilibrium strategy has a mass point at any strictly positive bid. Then, there must exist an interval $[\underline{t}, \bar{t}\,] \subseteq (0, \bar{s}_2]$ such that $[\underline{t}, \bar{t}\,] \subseteq \bigcap_{i \in I} supp_i$, $F_2(\underline{t}) < F(\underline{t})$, and $F_2(\bar{t}) > F(\bar{t})$. $[\underline{t}, \bar{t}\,] \subseteq supp_2$. Therefore, player 2 must earn his expected equilibrium payoff at any bid $x \in [\underline{t}, \bar{t}\,]$; that is, for every $x \in [\underline{t}, \bar{t}\,]$

$$\begin{aligned} u_2^*(x, F, F) &= v_{22}[F(x)]^2 + v_{2j}\left(1 - [F(x)]^2\right) - x \\ &= v_{2j} + \alpha [F(x)]^2 - x \\ &= v_{2j} + \alpha [F(0)]^2, \end{aligned}$$

where the last equality follows from Lemma A.1. Hence,

$$F(x) = \left([F(0)]^2 + \frac{x}{\alpha}\right)^{\frac{1}{2}} \text{ for all } x \in [\underline{t}, \bar{t}\,]. \tag{A.1.4}$$

Similarly, $[\underline{t}, \bar{t}\,] \subseteq supp_j, j \in \{1, 3\}$, implies that player j, $j \in \{1, 3\}$, must earn his expected equilibrium payoff at any bid $x \in [\underline{t}, \bar{t}]$. Player j's expected payoff from a bid, $x \in [\underline{t}, \bar{t}]$, is

$$u_j^*(x, F_2, F) = v_{jk} - \gamma \int_x^{\bar{s}_2} f_2(s)F(s)ds + \beta F(x)F_2(x) - x.$$

Player j's payoff must be constant on $[\underline{t}, \bar{t}]$, that is,

$$\frac{du_j^*(x)}{dx} = \gamma F_2'(x)F(x) + \beta\left(F_2'(x)F(x) + F_2(x)F'(x)\right) - 1 = 0 \text{ for all } x \in [\underline{t}, \bar{t}].$$

This yields the following linear first order differential equation, which must hold for all $x \in [\underline{t}, \bar{t}]$

$$F_2'(x)F(x)\alpha + F_2(x)F'(x)\beta = 1. \tag{A.1.5}$$

Since F takes the form described in (A.1.4), the solution to (A.1.5) is

$$F_2(x) = \frac{2\alpha}{\alpha + \beta}F(x) + c \cdot [F(x)]^{-\frac{\beta}{\alpha}},$$

where $c \in \mathbb{R}$ is a constant of integration.

By assumption $\beta > \alpha$, thus there exists a $\delta > 0$ such that $\beta = (1+\delta)\alpha$ and we can write

$$F_2(x) = \underbrace{\frac{2}{2+\delta}}_{<1} F(x) + c \cdot \underbrace{[F(x)]^{-(1+\delta)}}_{>0} \tag{A.1.6}$$

By differentiating (A.1.6) we obtain

$$F_2'(x) = \frac{2}{2+\delta}F'(x) - c \cdot \underbrace{(1+\delta)}_{>0}\underbrace{F'(x)}_{>0}\underbrace{[F(x)]^{-(2+\delta)}}_{>0}. \tag{A.1.7}$$

Suppose $F(\bar{t}) < F_2(\bar{t})$, then by continuity of the equilibrium strategies (Lemma A.2) $F(\bar{t}-\epsilon) < F_2(\bar{t}-\epsilon)$ for sufficiently small $\epsilon > 0$. Considering $x = \bar{t} - \epsilon$ in (A.1.6) yields the necessary condition $c > 0$. Using this in (A.1.7) shows that $F_2'(x) < F'(x)$ for $x \in [\underline{t}, \bar{t}]$. Hence, $F(\underline{t}) < F_2(\underline{t})$, a contradiction to the assumption that $F(\underline{t}) > F_2(\underline{t})$. Therefore, there exists no point $b_0 \in [0, \bar{s}_2]$ such that $F_2(b_0) < F(b_0)$, and F first order stochastically dominates F_2. □

Lemma A.10. $F_2(x) = 1$ *for all* $x \geq 0$

Proof. Lemmas A.1, A.8, and A.9 together imply $F(0) = 0$, hence by Lemma A.1 player 2's expected payoff in equilibrium is v_{2j}. By way of contradiction assume that $\bar{s}_2 > 0$. Then, by the same argument as in the proof of Lemma A.9 equation (A.1.6) must hold at every $x \in supp_2$ with $F(x) = \left(\frac{x}{\alpha}\right)^{\frac{1}{2}}$. Player 2 may not randomize over strictly positive bids arbitrarily close to zero. Indeed, if such randomization did occur, because all players' equilibrium strategies are continuous over $(0, \bar{s}]$ by Lemma A.2, $F(0) = 0$ and therefore

$$\lim_{\epsilon \to 0} F(\epsilon)^{-(1+\delta)} = \infty,$$

and $F_2(0) < 1$ (under the assumption that $\bar{s}_2 > 0$), then (A.1.6) would imply that $c = 0$, which is a contradiction to Lemma A.9. Given that player 2 does not randomize over strictly positive bids arbitrarily close to zero, there exists a $\underline{t} > 0$ such that $\underline{t} = \inf\{t > 0 | t \in supp_2\}$. Then, $F_2(\underline{t}) = F_2(0)$. By (A.1.1) player j's expected payoff from a bid $x \in (0, \underline{t}]$ is

$$u_j^*(x, F_2, F) = v_{jj} - \gamma \int_x^{\bar{s}} F(y) f_2(y) dy - \beta(1 - F_2(0)F(x)) - x.$$

$x \in (0, \underline{t}]$ is a best response for player j therefore $u_j^*(x, F_2, F)$ must be constant over $(0, \underline{t}]$. It follows that $F'(x) = \frac{1}{F_2(0)\beta}$ for $x \in (0, \underline{t}]$. From $F(0) = 0$ follows that players 1 and 3 randomize uniformly over $[0, \underline{t}]$ according to

$$F(x) = \frac{x}{F_2(0)\beta}, \, x \in [0, \underline{t}].$$

Continuity of F at $\underline{t}$ yields

$$\frac{t}{F_2(0)\beta} = \left(\frac{\underline{t}}{\alpha}\right)^{\frac{1}{2}} \Leftrightarrow \underline{t} = \frac{\beta^2}{\alpha} \cdot [F_2(0)]^2.$$

Consequently,

$$F(\underline{t}) = (1+\delta)F_2(0).$$

Using this and $F_2(\underline{t}) = F_2(0)$ in (A.1.6) yields

$$\begin{aligned} F_2(0) &= F_2(\underline{t}) \\ &= \frac{2}{2+\delta} F(\underline{t}) + c \cdot [F(\underline{t})]^{-(1+\delta)} \\ &= \frac{2}{2+\delta}(1+\delta)F_2(0) + c \cdot [(1+\delta)F_2(0)]^{-(1+\delta)}, \end{aligned}$$

which implies

$$c = \left(-\frac{\delta}{2+\delta}F_2(0)\right)[(1+\delta)F_2(0)]^{1+\delta} \leq 0.$$

This contradicts Lemma A.9; therefore $\bar{s}_2 = 0$. □

Altogether, this shows that player 2 stays out of the conflict in equilibrium. Hence, the equilibrium described in Proposition 2 is the unique equilibrium of Γ_{21}. □

A.2 Proof of Proposition 6

Proof. Under the assumption that all three players make positive bids with strictly positive probability and players 1 and 3 use identical strategies, i.e. $F_1 = F_3 =: F$, we know that $\underline{s}_1 = \underline{s}_2 = \underline{s}_3 = 0$ and $\bar{s}_2 = \bar{s}_1 = \bar{s}_3 =: \bar{s}$. Moreover, $\bar{s} \in (r_{jk}, r_{2j})$, $j,k \in \{1,3\}, j \neq k$, and player 2 cannot have a masspoint at zero. Assume that all players randomize continuously over $[0,\bar{s}]$. All players must earn their equilibrium payoff at $\bar{s}$, therefore player 2's expected payoff from a bid $b \in (0, \bar{s}]$, $u_2(b, F) = v_{22}[F(b)]^2 + v_{2i}(1 - [F(b)]^2)$, must be $v_{22} - \bar{s}$. This yields

$$F(x) = \begin{cases} 0 & x < 0 \\ \left[\left(1 - \frac{\bar{s}}{r_{2j}}\right) + \frac{x}{r_{2j}}\right]^{\frac{1}{2}} & 0 \le x \le \bar{s} \\ 1 & x > \bar{s} \end{cases} .$$

Player j's payoff must be $v_{jj} - \bar{s}$. Moreover, player j chooses his equilibrium strategy such that his expected payoff, $u_j(b, F_2, F) = -b + v_{j2} + [v_{jj} - v_{j2}]F(b)F_2(b) + [v_{jk} - v_{j2}] \int_b^{\bar{s}} F_2(s)F'(s)ds$, is maximized. The first order condition yields the first order differential equation

$$0 = F(x)F_2'(x)r_{j2} + F'(x)F_2(x)r_{jk} - 1.$$

Using the boundary conditions $F_2(0) = 0$ and $F_2(\bar{s}) = 1$ this yields

$$F_2(x) = \kappa F(x) - (\kappa - 1)F(x)^{-\frac{r_{jk}}{r_{j2}}}$$

with $\kappa = \frac{2r_{j2}}{r_{j2}+r_{jk}} > 1$ and $\bar{s} = r_{j2}\left[1 - \left(1 - \frac{1}{\kappa}\right)^{\kappa}\right]$. Note that $\bar{s} \in (r_{jk}, r_{j2})$ and F_2 is strictly increasing.

In order to show that this equilibrium exhibits extremism, we need to show that $F_2(x) \le F(x)\ \forall x$. All players' cdfs coincide for $x < 0$ and $x \ge \bar{x}$. The centrist players put strictly positive mass on zero, thus $F_2(0) < F(0)$. For $x \in (0, \bar{x})$,

$$F_2(x) = \kappa F(x) - (\kappa - 1)F(x)^{-\frac{r_{13}}{r_{12}}} = F(x) \underbrace{\left[\kappa - (\kappa - 1)\overbrace{F(x)^{-\left(1 + \frac{r_{13}}{r_{12}}\right)}}^{>1}\right]}_{<\kappa-(\kappa-1)=1} < F(x).$$

Therefore, F_2 first order stochastically dominates F. □