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# Why Uncertainty Matters – Discounting under Intertemporal Risk Aversion and Ambiguity

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## Abstract

Uncertainty has an almost negligible impact on project value in the economic standard model. I show that a comprehensive evaluation of uncertainty and uncertainty attitude changes this picture fundamentally. The analysis relies on the discount rate, which is the crucial determinant in balancing immediate costs against future benefits and the single most important determinant of optimal mitigation policies in the integrated assessment of climate change. The paper examines two shortcomings in the recent debate and the current models addressing climate change assessment. First, removing an implicit assumption of (intertemporal) risk neutrality reduces the growth effect in social discounting and significantly amplifies the importance of risk and correlation. Second, debate and models largely overlook the difference in attitude with respect to risk and with respect to non-risk uncertainty. The paper derives the resulting changes of the risk-free and the stochastic social discount rate and points out the importance of even thin tailed uncertainty for climate change evaluation. It discusses combinations of uncertainty and correlation that reduce the social discount rate to pure preference. In a theoretical contribution, the paper extends the smooth ambiguity model by providing a threefold disentanglement between, risk aversion, ambiguity aversion, and the propensity to smooth consumption over time.


JEL-Code: D610, D810, D900, H430, Q000, Q540.

Keywords: ambiguity, climate change, cost benefit analysis, discounting, intertemporal substitutability, risk aversion, uncertainty.

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# 1 Introduction

## 1.1 Overview

Most long-term investments are subject to uncertainty. In the standard model, uncertainty has an almost negligible impact on project value. I show that a comprehensive evaluation of uncertainty and uncertainty attitude changes this picture fundamentally. The theoretic contribution of the present paper combines intertemporal risk attitude with smooth ambiguity aversion. Intertemporal risk attitude is a multi-commodity risk measure that captures Epstein & Zin's (1989) disentanglement of Arrow-Pratt risk aversion from the propensity to smooth consumption over time. Smooth ambiguity attitude is a concept developed by Klibanoff, Marinacci & Mukerji (2005) to capture aversion to non-risk uncertainty, i.e. beliefs of low confidence. I show and exploit the close formal similarity of the two concepts.

The paper derives and discusses the consumption discount rate in this generalized uncertainty model. The consumption discount rate determines the optimal trade-off between current investment costs and future investment payoffs. In the context of public projects, this consumption discount rate is known as the social discount rate. The U.K. and France were the first countries to explicitly adapt their discounting schemes for the evaluation of legislation and projects to recognize uncertainty. The U.S. Environmental Protection Agency is currently preparing a similar proposal to the Office of Management and Budget. The economic insights underlying these political reforms are all based on the economic standard model. However, it is well-known that this standard model gives rise to a variety of puzzles in asset pricing under uncertainty, including the equity premium and the risk free rate puzzles. These puzzles are easily explained by more comprehensive uncertainty models. In particular, the finance literature shows and exploits the fact that agents are more risk averse than they are averse to consumption substitution in the time dimension: they are intertemporal risk averse. I demonstrate the importance of this risk attitude for discounting and long-term evaluation.

Growth uncertainty alone implies a major adjustments of the social discount rate. However, many projects have uncertain payoffs themselves. I show that correlation between project payoffs and economic baseline growth become highly relevant for project value once discount rates incorporate general risk attitude. Examples of large scale projects (or legislation) with uncertain payoffs include investments into basic research, national defense, development of new energy technologies, or climate change adaptation and mitigation. I focus my application on the latter example, a field where the social discount rate is currently most hotly debated (Stern 2007, Nordhaus 2007, Weitzman 2007, Weitzman 2009, Dasgupta 2009, Heal 2009). As pointed out by Nordhaus (2007), the social discount rate is the single most important explanatory variable when comparing the optimal policy recommendation across different integrated assessments of climate changes: a social discount rate of 1.4% as chosen in the Stern (2007) review implies an optimal present day carbon tax that is 10 times higher than using the 5.5% used by Nordhaus (2008). Almost all large scale integrated assessment models are based on the economic standard model. I show that in models of comprehensive risk attitude uncertainty results in adjustments of the social discount rate in the same order of magnitude. Moreover, in the climate context, there is support for positive, negative and for no correlation between mitigation payoffs and economic growth. I find that correlation is highly relevant to optimal climate policy and, thus, the paper calls for a more careful analysis which correlation channel dominates in the climate debate.

The paper focuses on analytic extensions on the Ramsey rule with simple tractable solution. These formulas can be employed for applied evaluation. They also invite back of the envelope calculations determining the relative importance of the different discounting contributions. Accounting for uncertainty in large scale economic models is often computationally expensive. Then, these formulas guide a cost benefit analysis of model extensions. In order to derive analytic results, I restrict attention to isoelastic preferences and normal uncertainty. This preference restriction also enables me to use estimates from the asset pricing literature to compare the magnitudes of the different terms in the extended Ramsey equation (Vissing-Jørgensen & Attanasio 2003, Basal & Yaron 2004, Basal, Kiku & Yaron 2010).

For my quantitative analysis I focus on the case of risk, where parametric estimates have converged to a more reliable quantification of general attitude. However, I show that the analysis carries over immediately to the case of smooth ambiguity aversion. I point out a striking analogy between intertemporal risk aversion measuring aversion to confidently known (or objective) risk and smooth ambiguity aversion measuring aversion to subjective second order uncertainty. I show that, in consequence, the adjustments of the social discount rate to ambiguity attitude are almost identical to the adjustments in the case of intertemporal risk aversion. A major theoretical contribution of this paper is a model that disentangles threefold between risk attitude, consumption smoothing attitude, and smooth ambiguity aversion. I discuss the implications of the threefold disentanglement for the discount rate, and employ the ambiguity layer to analyze the impact of a low confidence growth expectation as well as an 6ignorant prior over correlation.

## 1.2 Related Literature

The paper combines preferences disentangling risk attitude from the propensity to smooth consumption over time with the smooth ambiguity model of Klibanoff et al. (2005). The disentanglement of aversion to risk and aversion to intertemporal substitution goes back to Selden (1978), Kreps & Porteus (1978), Epstein & Zin (1989), and Weil (1990). Traeger (2010) extends the Epstein-Zin-Weil disentanglement to a multi-commodity setting introducing the concept of intertemporal risk aversion. The main insight of these models can be summarized as follows. Risk aversion generally has two effects on evaluation. First, stochasticity generates wiggles in the consumption path. Agents with a propensity to smooth consumption over time dislike these wiggles. This risk effect, and only this risk effect, is captured in the intertemporally additive expected utility standard model. Second, agents intrinsically dislike risk because it creates uncertainty about the future. This second effect is measured by intertemporal risk aversion (and is good independent). As opposed to a widespread believe, the von Neumann & Morgenstern (1944) axioms support intrinsic risk aversion (Traeger 2010). In the one commodity model of Epstein & Zin (1989) and Weil (1990) the disentangled Arrow Pratt measure jointly captures risk aversion as generated by both, 'wiggles' and intrinsic risk aversion. The smooth ambiguity model of Klibanoff et al. (2005) and Klibanoff, Marinacci & Mukerji (2009) does not capture intrinsic risk aversion. I combine the smooth ambiguity model with the disentanglement of risk aversion from the propensity to smooth consumption over time. I show that the measure of smooth ambiguity aversion is an analogue to the measure of intertemporal risk aversion.

Gollier (2002) shows in a model disentangling risk aversion from intertemporal consumption smoothing a crucial condition for the social discount rate to decrease under risk: the disentangled Arrow Pratt measure of absolute risk aversion has to decrease in consumption. This finding closely relates to Leland's (1968) condition stating that savings increase under uncertainty if entangled Arrow Pratt risk aversion decreases in consumption. The condition is widely believed to hold and, in particular, it is satisfied in case of isoelastic preferences employed in the current paper. Apart from adding ambiguity, the present paper complements and extends Gollier (2002) analysis in several ways. First, I use quantitative estimates from the asset pricing literature to discuss the magnitude of the various contributions to the social discount rate. Second, my formulation in terms of intertemporal risk aversion collects different contributions from Gollier's (2002) into a single simple-to-interpret adjustment of the discount rate proportional to intertemporal risk aversion. Third, this reformulation in terms of intertemporal risk aversion shows that Epstein-Zin preferences imply a largely equivalent adjustment of the discount rate as does smooth ambiguity aversion. Fourth, I extend the setting to account for the important correlation between project payoffs and baseline uncertainty.

Paralleling the current research, Gierlinger & Gollier (2008) analyze the social discount rate in Klibanoff et al.'s (2005) smooth ambiguity framework. While I focus on the analytic extension of the Ramsey formula and discuss the magnitude and relevance of individual terms, Gierlinger & Gollier (2008) focus on general ambiguity attitude and qualitative characterizations of the impact of uncertainty and ambiguity aversion. In comparison to their paper, I sacrifice generality of functional forms for the sake of analytic tractability and generality in terms of conceptual extensions. First, the current paper adds stochasticity of the investment projects and shows how general uncertainty attitude amplifies the importance of the correlation between economic baseline growth and the stochastic payoffs of the project. Second, Klibanoff et al. (2005) and, thus, Gierlinger & Gollier (2008) conflate the disentanglement between the propensity to smooth consumption over time versus risk states with aversion to ambiguity.[1] In contrast, I provide a clear threefold disentanglement of all three preference dimensions. A more minor difference is that my derivation of the social discount rate does not rely on an equilibrium assumption. In the climate change context, for example, mitigation efforts are not close to an efficient Lucas (1987) tree model equilibrium as described in Gierlinger & Gollier (2008).

Weitzman (2007, 2009) argues that uncertainty gives rise to a low social discount rate in climate change assessment. Weitzman reaches this conclusion by following a Bayesian approach to modeling structural uncertainty. His analysis delivers a fat-tailed posterior over damages that translates into a high willingness to pay for a (certain) transfer into the future.[2] Instead of following Weitzman's path of augmenting uncertainty to somewhat contestable levels, I follow the decision theoretic developments that treat uncertainty attitude more comprehensively. Finally, following the original working paper version of this paper, Ju & Miao (2009) put forth a calibration for a version of the three-fold disentangling model I present here.[3]

[1]In the original smooth ambiguity model, aversion to standard or objective risk is set equal to the propensity to smooth consumption over time. Only aversion to subjective risk, or second order uncertainty, is disentangled from this intertemporal smoothing preference. Thus, the original smooth ambiguity model conflates ambiguity aversion with well known risk characteristics: already objective risk aversion is usually larger than the propensity to smooth consumption intertemporally. Only introducing such a disentanglement for subjective uncertainty results in an unfair comparison between the effects of risk and ambiguity aversion.

[2]For critical discussions of this approach, see in particular Pindyck (2009), Nordhaus (2009), Horowitz & Lange (2009), and Millner (2011).

[3]They find a coefficient of smooth ambiguity aversion very close to the risk aversion coefficients I discuss in the context of risk aversion. However, their approach exogenously assumes a low value

Section 2 provides the background to the paper, discussing the consumption discount rate in the standard model and introducing the concept of intertemporal risk aversion. Section 3 extends the discounting formula to intertemporal risk aversion and stochastic projects. I apply the framework to analyze a stylized trade-off over time horizons relevant to the evaluation of mitigation and adaptation projects and show how, in a model of comprehensive risk attitude, already minor growth risk can reduce the social discount rate to pure time preference. Section 4 incorporates smooth ambiguity aversion and applies the model to second order uncertainty over expected growth and over the correlation between project payoff and baseline uncertainty. Section 5 concludes.

# 2 Background

## 2.1 Discounting the Future under Uncertain Growth

The consumption or social discount rate characterizes how the value of consumption develops over time. This section lays out the basic setting and summarizes important aspects of the recent debate over the correct social discount rate (Stern 2007, Nordhaus 2007, Weitzman 2007, Weitzman 2009, Dasgupta 2009, Plambeck, Hope & Anderson 1997). First period consumption $x_1 \in X$ is certain while second period consumption is captured by the probability measure $p$ over $X$.[4] In the *standard model*, a decision maker evaluates utility for every period and for every state of the world by a utility function $u$ and sums over states and over time to arrive at

$$U^s(x_1, p) = u(x_1) + e^{-\delta} \mathrm{E}_p u(x_2) \ . \tag{1}$$

The utility discount rate $\delta$ is known as the rate of pure time preference.

The decision maker faces a trade-off between aggregate consumption in the present and in the future. Growth is stochastic and his utility is $u(x) = \frac{x^{1-\eta}}{1-\eta}, \eta > 0, \eta \neq 1$. Given $x_1$, the consumption growth rate $g = \ln \frac{x_2}{x_1}$ is normally distributed with $g \sim N(\mu, \sigma^2)$. The *risk-free social or consumption discount rate* $r = \ln \frac{dx_2}{-dx_1}|_{\bar{U}}$ characterizes a marginal certain trade-off between the future ($dx_2$) and the present ($dx_1$) that leaves overall welfare unchanged:

$$r = \delta + \eta\mu - \eta^2 \frac{\sigma^2}{2} \ . \tag{2}$$

of Arrow-Pratt (and, thus, intertemporal) risk aversion. Then, their coefficient of relative smooth ambiguity aversion picks up the remaining aversion necessary to explain asset prices.

[4]Formally, $X$ is a compact metric space and $p \in P$ an element of the space of Borel probability measures on $X$.

The formula is a well-known extension of the classic Ramsey (1928) formula that makes growth stochastic. More precisely, the consumption or social discount rate is concerned with the right hand side of the Ramsey equation. Given incomplete markets, externalities, and transaction costs, this right hand side is a preferred measure for optimality of trade-offs characterizing long-term projects and legislation that affect consumption. I emphasize the *risk-free* nature of the trade-off characterized by the consumption (or social) discount rate. The rate is good for evaluating deterministic projects in an uncertain growth scenario. Frequently, the rate in equation (2) is also applied to evaluate certainty-equivalent project payoffs, however, section 3.2 discusses why for stochastic projects the stochastic social discount rate should be used instead. While the first term characterizing the discount rate reflects pure impatience, the second term is a consequence of economic growth. The consumption elasticity of marginal utility $\eta$ characterizes the percentage decrease in marginal utility from a percentage increase of consumption. Together with the expected growth rate $\mu$, the term $\eta\mu$ characterizes the decrease of marginal utility because of expected consumption growth. The parameter $\eta$ captures aversion to intertemporal consumption changes. These changes include expected changes captured in the term proportional to $\mu$ and fluctuations generating wiggles in the consumption path captured in the term proportional to $\sigma^2$. This aversion to wiggles in the consumption path corresponds to the only risk contribution in the standard model and $\eta$ is simultaneously interpreted as a measure of risk aversion. For the annual discount rate, the parameters $\delta, \mu$, and $\sigma$ are in the order of percent, while $\eta$ is in the unit order. Therefore, $\sigma^2$ makes the third term $10-100$ times smaller than the others and risk can be neglected in discounting.[5]

The parameter choices of Stern (2007) can be approximated by $\delta = 0.1\%$, $\eta = 1$, and $\mu = 1.3\%$ delivering $r = 1.4\%$ under certainty. While Stern's team argues from a normative perspective for these choices, the majority of integrated assessment modelers refuses such a standpoint.[6] A representative of this positive school is Nordhaus, creator of the widespread open-source integrated assessment model DICE. His parameter choices in the recent version DICE-2007 (Nordhaus 2008) are $\delta = 1.5\%$, $\eta = 2$, and $\mu = 2\%$[7] delivering $r = 5.5\%$ (again under certainty). I already pointed out that this difference in the social discount rate implies a factor 10 difference in the resulting optimal carbon

[5] Be aware that $\sigma$ characterizes risk in the sense of volatility. In the climate change debate, risk is frequently used to also denote a reduction in the expected value, e.g. as a consequence of catastrophic events. Such a reduction would mostly affect the expected growth term of the social discount rate.

[6] Moreover, Dasgupta (2008) points out that, from a normative perspective, an egalitarian choice of $\delta = 0.1\%$ should also call for a higher propensity of intergenerational consumption smoothing $\eta > 1$.

[7] The growth rate is endogenous in the DICE model and has been reconstructed from Nordhaus (2007, 694).

tax. Introducing uncertainty with a standard deviation of $\sigma = 2\%(4\%)$ results in an adjustment of the risk-free rate by 0.02%(0.08%) in the case of Stern and 0.08%(0.3%) in the case of Nordhaus. The lower standard deviation of $\sigma = 2\%$ is used by Weitzman (2009) to approximate the volatility of economic growth without climate change and catastrophic risks. The high standard deviation of $\sigma = 4\%$ is the rounded estimate of historic consumption fluctuations by Kocherlakota (1996).[8] The values for the low standard deviation are negligible, while the high standard deviation results in minimal adjustments.

## 2.2 Intertemporal Risk Aversion

The standard model of the previous section implicitly assumes that a decision maker's aversion to risk coincides with his aversion to intertemporal variation. Epstein & Zin (1989) and Weil (1990) derive an alternative setting that disentangles these two a priori quite different characteristics of preference. Traeger (2010) extends their framework to a multi-commodity setting. For this purpose, the author introduces a new measure of *intertemporal risk aversion* that measures the difference between Arrow-Pratt risk aversion and the propensity to smooth consumption over time. The current section motivates the Epstein-Zin generalization of risk attitude along the lines of intertemporal risk aversion. Later sections use of the intertemporal risk aversion measure to give a more compact characterization of the social discount rate adjustment under general risk attitude and to show and exploit its similarity to the smooth ambiguity measure in the case of general uncertainty. From the perspective of intertemporal risk aversion the standard model is risk neutral. It only generates aversion to stochasticity because of the wiggles in the consumption path and not because of intrinsic aversion to being uncertain about the future. As a consequence, expressing the social discount rate in terms of intertemporal risk aversion splits its constituents cleanly into those contributions arising in the standard model and those additional contributions that are due to intertemporal risk aversion.

The curvature of the utility function $u$ in equation (1) captures both, aversion to risk and aversion to intertemporal variation. A priori, however, risk aversion and the propensity to smooth consumption over time are two distinct concepts. More generally, welfare is characterized by two independent functions corresponding to these two distinct

[8]Kocherlakota (1996) estimates $\mu = 1.8\%$ and $\sigma = 3.6\%$ based on 90 years of annual data for the US.

preference characteristics

$$U(x_1, p) = u(x_1) + e^{-\delta} f^{-1} \left[ \mathrm{E}_p f \circ u(x_2) \right] . \tag{3}$$

Here, utility $u$ is a measure for the appreciation of a particular consumption bundle that derives from the willingness to trade over time. The concavity of $u$ captures the aversion to intertemporal consumption variation. The curvature of $f$ describes intertemporal risk aversion, which can be interpreted as aversion with respect to utility gains and losses. Note that the curvature of $f$ is a one-dimensional risk measure even in a multi-commodity world.[9] Opposed to a widespread believe, equation (3) – not equation (1) – is the general representation of preferences satisfying the von Neumann & Morgenstern (1944) axioms, additive separability over certain consumption paths, time consistency, and (finite time) stationarity (Traeger 2007).[10]

A representation-free characterization of intertemporal risk aversion motivates why the standard model generally falls short of capturing risk attitude comprehensively. The general definition is provided in Traeger (2010). It requires at least two uncertain periods. Here, I give a simplified characterization that, however, requires the absence of pure time preference.[11] Let $\succeq$ characterize preferences on $X \times P$ representable by equation (3) with $\delta = 0$. A decision maker is called (weakly)[12] intertemporal risk averse, if and only if, for all $x^*, x_1, x_2 \in X$

$$(x^*, x^*) \sim (x_1, x_2) \quad \Rightarrow \quad (x^*, x^*) \;\; \succeq \;\; \left(x^*, (\tfrac{1}{2}, x_1; \tfrac{1}{2}, x_2)\right) , \tag{4}$$

where the term $(\frac{1}{2}, x_1; \frac{1}{2}, x_2)$ characterizes a fair coin flip returning either $x_1$ or $x_2$. The premise in equation (4) states that a decision maker is indifferent between a certain constant consumption path delivering the same outcome $x^*$ in both periods and another certain consumption path that delivers outcome $x_1$ in the first and outcome $x_2$ in the second period. For example, $x_1$ can be an inferior outcome with respect to $x^*$. Then, $x_2$ is a superior outcome with respect to $x^*$. On the right-hand side of equation (4),

[9]See Kihlstrom & Mirman (1974) for the complications that arise when trying to extend the Arrow Pratt risk measures to a multi-commodity setting. Even more interestingly, measures of intertemporal risk aversion can be applied straightforwardly to contexts where impacts do not have a natural cardinal scale.

[10]Note that, in general, preferences represented by equation (3) cannot be represented by an evaluation function of the form $U^s(x_1, p) = u_1(x_1) + \mathrm{E}_p u_2(x_2)$.

[11]I abandon pure time preference for the sake of simplicity in the characterization only. This step does not change the intuition of the axiom with respect to its general form. Obviously, I keep pure time preference when discussing discount rates.

[12]The strong notion would involve the additional requirement $(x^*, x_1) \not\sim (x^*, x_2)$ in the premise and a strict preference in the implication.

the decision maker receives $x^*$ in the first period, independent of his choice. For the second period, he has a choice between the certain outcome $x^*$ or a lottery that returns with equal probability either the superior or the inferior outcome. The decision maker is called (weakly) intertemporal risk averse if he prefers the certain outcome $x^*$ in the second period over the lottery.[13] I show in Proposition 5 in Appendix B that a decision maker (with $\delta = 0$) is intertemporal risk averse in the sense of equation (4) if and only if the function $f$ in the representation (3) is concave.

In the two period setting, the intertemporally additive reformulation of Epstein & Zin's (1989, 1991) infinite horizon recursive utility model is[14]

$$U(x_1, p) = \frac{x_1^{1-\eta}}{1-\eta} + e^{-\delta} \frac{1}{1-\eta} \left[\mathrm{E}_p x_2^{1-\mathrm{RRA}}\right]^{\frac{1-\eta}{1-\mathrm{RRA}}} , \tag{5}$$

where RRA is the coefficient of Arrow Pratt risk aversion. It is easily verified that equation (5) results from equation (3) using the intertemporal risk aversion function

$$f(z) = \big((1-\eta)z\big)^{\frac{1-\mathrm{RRA}}{1-\eta}} . \tag{6}$$

Instead of Arrow-Pratt risk aversion, I will make use of the measure of relative intertemporal risk aversion

$$\mathrm{RIRA}(z) = -\frac{f''(z)}{f'(z)}\,|z| = \begin{cases} 1 - \frac{1-\mathrm{RRA}}{1-\eta} \text{ if } \eta < 1 \\ \frac{1-\mathrm{RRA}}{1-\eta} - 1 \text{ if } \eta > 1 \,. \end{cases} \tag{7}$$

[13]The lottery on the right-hand side of equation (4) will either make the decision maker better off or worse off than $(x^*, x^*)$, while, on the left-hand side, the decision maker knows that if he picks an inferior outcome for some period he certainly receives the superior outcome in the other.

Calling preferences satisfying equation (4) intertemporal risk averse is motivated by the facts that, first, the definition intrinsically builds on intertemporal trade-offs and, second, Normandin & St-Amour (1998, 268) make the point that the conventional Arrow Pratt measure of risk aversion is an atemporal concept.

A decision maker is defined as (weakly) intertemporal risk loving if the preference relation $\succeq$ in equation (4) is replaced by $\preceq$. He is defined to be risk neutral if he is both intertemporal risk loving and intertemporal risk averse (relation $\succeq$ in equation 4 is replaced by $\sim$).

[14]In a multiperiod framework equation (5) translates into the recursion

$$U(x_{t-1}, p_t) = \frac{x_{t-1}^\rho}{\rho} + \beta \frac{1}{\rho} \left[\mathrm{E}_{p_t} \left(\rho U(x_t, p_{t+1})\right)^{\frac{\alpha}{\rho}}\right]^{\frac{\rho}{\alpha}} , \tag{$\star$}$$

To obtain the normalization used by Epstein & Zin (1989, 1991), multiply equation ($\star$) by $(1-\beta)\rho$ and take both sides to the power of $\frac{1}{\rho}$. Define $U^*(x_{t-1}, p_t) = ((1-\beta)\rho U(x_{t-1}, p_t))^{\frac{1}{\rho}}$. Expressing the resulting transformation of equation ($\star$) in terms of $U^*$ delivers their version

$$U^*(x_{t-1}, p_t) = \Big((1-\beta) x_{t-1}^\rho + \beta \left[\mathrm{E}_{p_t} \left(U^*(x_t, p_{t+1})\right)^\alpha\right]^{\frac{\rho}{\alpha}}\Big)^{\frac{1}{\rho}} .$$

The measure RIRA($z$) depends on the choice of zero in the definition of the utility function $u$. This normalization-dependence is the analog to e.g. the wealth level dependence of the Arrow Pratt measure of relative risk aversion.[15][16] Traeger (2010) further elaborates that $f$ and RIRA can be interpreted as a measure for the difference between Arrow Pratt risk aversion and the willingness to smooth consumption over time.

Estimates of the isoelastic model usually build on Epstein & Zin (1991) and Campbell's (1996) log-linearizing the Euler equations. The estimation of the isoelastic model is significantly more challenging than in the case of the standard model.[17] However, over the recent years a somewhat reliable set of parameters emerges to be $\eta = \frac{2}{3}$ and RRA $\in [8, 10]$, explaining well asset prices and related puzzles (Vissing-Jørgensen & Attanasio 2003, Basal & Yaron 2004, Basal et al. 2010). The message of these estimates and calibration results is that agents tend to have a higher aversion to risk than to intertemporal substitution. In contrast, the standard model forces both parameters to coincide and the joint, entangled estimate falls somewhere in-between. For my quantitative analysis, I use the entangled standard model with $\eta = 2$ as scenario "N". The value of 2 is widespread and, in particular, employed in Nordhaus's (2008) integrated assessment of climate change. The standard model implies zero intertemporal risk aversion. For the disentangled model "D", I use the values $\eta = \frac{2}{3}$ and RRA $= 9.5$ singled out by Vissing-Jørgensen & Attanasio (2003). These estimates imply a coefficient of relative intertemporal risk aversion of RIRA $= 27$. Depending on the assessment, I also provide sensitivity scenarios or vary parameters on a continuum.

# 3 Discounting Under Intertemporal Risk Aversion

## 3.1 Risk Free Projects

Intertemporal risk aversion results in the following adjustment of the risk-free rate.

[15] In the standard model, the Arrow Pratt measure of relative risk aversion depends on what is considered the $x = 0$ level. For example, whether or not breathing fresh air is part of consumption or whether human capital is part of wealth changes the Arrow Pratt coefficient.

[16] Note that positivity of RIRA indicates intertemporal risk aversion independently of whether $f$ is increasing and concave or decreasing and convex (see footnote 30). In both cases $-\frac{f''}{f'}$ is positive. Moreover, measuring utility in negative units as in the isoelastic case for $\rho < 0$ makes $z$ negative. Therefore, the definition of relative risk aversion has to employ the absolute of the variable $z$ (Traeger 2010). The same reasoning applies to the measure of smooth ambiguity aversion.

[17] These models have to make assumptions about the covariance of consumption growth and stock returns, the share of stocks in the financial wealth portfolio, the properties of the expected returns to human capital, and the share of human capital in overall wealth.

**Proposition 1:** The risk-free social/consumption discount rate in the isoelastic setting with intertemporal risk aversion is

$$r = \delta + \eta\mu - \eta^2 \frac{\sigma^2}{2} - \text{RIRA } \left|1-\eta^2\right| \frac{\sigma^2}{2} \,. \tag{8}$$

In the presence of growth uncertainty, a decision maker exhibiting positive intertemporal risk aversion RIRA $> 0$ discounts the future payoffs at a lower rate. In consequence, an intertemporal risk averse decision maker is willing to invest in certain projects with a relatively lower productivity than a decision maker who bases his decision on the standard model. Gollier (2002) has derived an analogue of equation (8). In his representation the single intertemporal risk averison term above is replaced by an Arrow Pratt risk aversion and a prudence term. The slightly simpler form that I employ in equation (8) has the advantage that it pinpoints the deviation between the standard model (RIRA $= 0$) and the generalized isolastic model. Moreover, this representation will demonstrate the formal similarity between the generalized isoelastic model and the smooth ambiguity model in section 4.

In equation (8), the parameter $\eta$ reflects only aversion to intertemporal fluctuations. Therefore, the term $\eta^2 \frac{\sigma^2}{2}$ is interpreted as the cost of expected fluctuations triggered by the aversion to non-smooth intertemporal consumption paths. I keep referring to the expression as "the standard risk term", as it is the only expression capturing risk in an analysis based on the standard model. In the case of fully disentangled preferences (scenario D, see section 2.2), the magnitude of the intertemporal risk aversion contribution is

$$\frac{\text{RIRA } |1-\eta^2| \frac{\sigma^2}{2}}{\eta^2 \frac{\sigma^2}{2}} \approx 33 \,,$$

times that of the standard risk contribution. Figure 1 depicts the different discounting contributions as a function of $\eta$. The graph keeps RRA $= 9.5$, $\mu_t = 2\%$ and $\sigma_t = 4\%$ fix.

The positive growth term (brown, dash-dotted) dominates for reasonably high values of $\eta$. The intertemporal risk aversion term (blue, dashed) defines the main reduction. The standard risk term (black, dotted) plays a very minor role in determining the overall discount rate net of pure time preference (green, solid). Note that the intertemporal risk aversion contribution is continuous at $\eta = 1$. Keeping RRA fix, RIRA is itself a function of $\eta$ as it measures the difference between Arrow-Pratt risk aversion and aversion

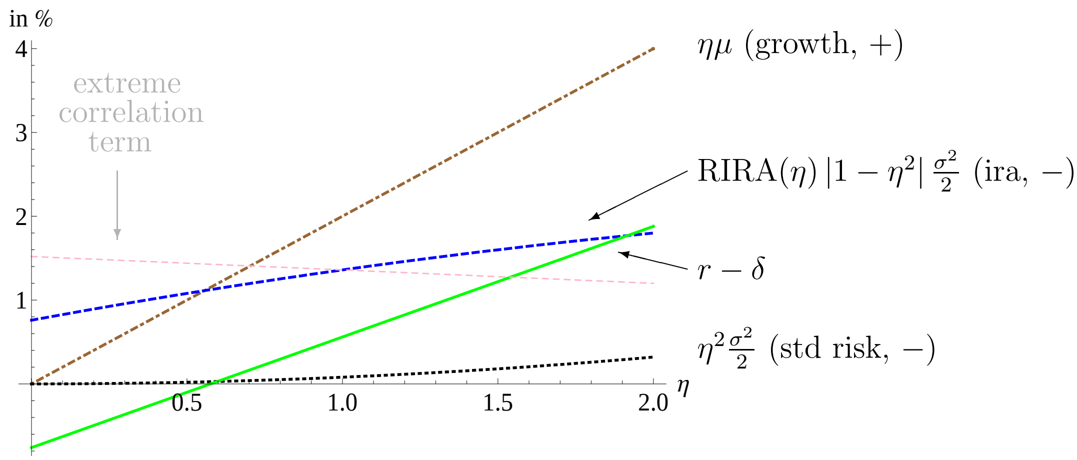


Figure 1 depicts the different contributions of the discount rate as well as the total discount rate net of pure time preference $r - \delta$. The terms are drawn as a function of aversion to intertemporal fluctuations $\eta$, keeping relative Arrow-Pratt risk aversion fixed at RRA = 9.5 and using $\mu_t = 2\%$ and $\sigma_t = 4\%$. A minus sign in the bracket implies that the term is negative and subtracted from the positive growth term. The abbreviation ira denotes the contribution from intertemporal risk aversion.

to intertemporal substitution.[18] Moving from the standard model with RRA $= \eta = 2$ to the disentangled model with $\eta = \frac{2}{3}$ implies two changes in the discount rate. First, the growth effect is significantly reduced once $\eta$ only captures attitude to intertemporal substitution (brown dash-dotted line is evaluated further to the left). Second, intertemporal risk aversion reduces the discount rate (blue dashed line now subtracted from the brown dashed line). This second effect is the direct effect of intertemporal risk aversion. The first one is an indirect effect: moving from $\eta = 2$ to $\eta = \frac{2}{3}$ it is even larger in magnitude than the direct effect.

Figure 2 fleshes out the crucial difference in the relation between risk and discount rates in the standard model and a model of general risk attitude. The standard model confines RRA $= \eta$. The thick colored line moving upwards from the origin depicts the discount rate net of pure time preference $r - \delta$ in the standard model. The yellow region of the otherwise red line reflects the most common preference specifications $\eta \in [1, 2]$. Accounting for higher risk aversion in the standard model moves $r - \delta$ up along the RRA $= \eta$ line and significantly increases the discount rate. In contrast, higher risk

[18]The Epstein-Zin preference representation in equation (5) implies a switch in the sign of utility when $\eta$ crosses unity. During this sign switch $1 - \eta$ goes through zero, while RIRA has a pole. One could redefine RIRA $|1 - \eta^2|$ as the actual measure of intertemporal risk aversion, as it is positive if and only if equation (4) holds. I stick to the definition in equation (7) because this measure is completely analogous to the measure suggested for smooth ambiguity aversion.

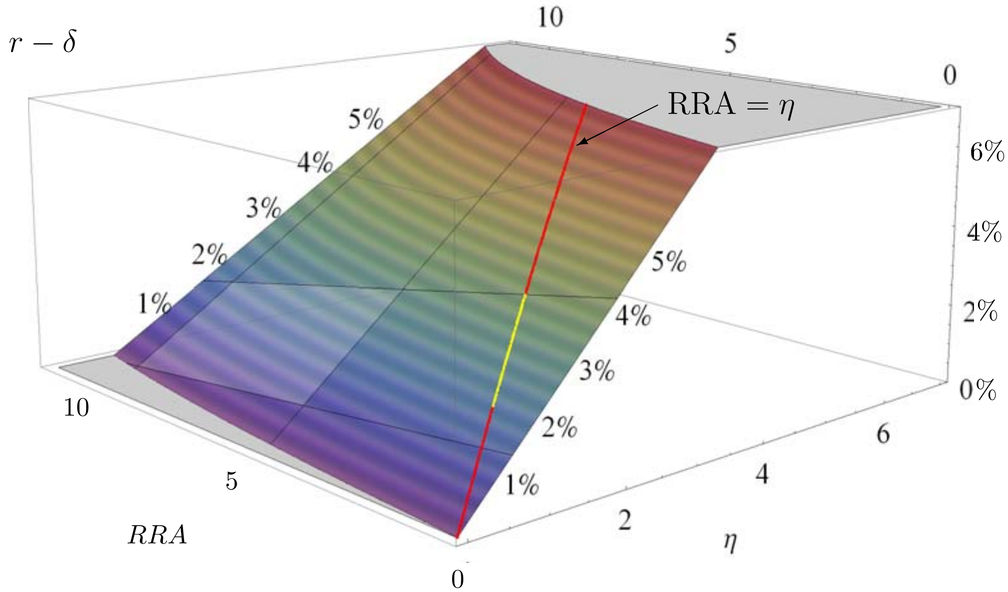


Figure 2 depicts the total discount rate net of pure time preference $r - \delta$ as a function of $\eta$ and RRA. Moving along the thick (red and yellow) line keeps RRA $= \eta$, representing the only movement possible in the standard model. The yellow part of the line marks the most common parameterization $\eta \in [1, 2]$. Increasing risk aversion along this line necessarily leads to high discount rates. In contrast, increasing risk aversion in the disentangled model corresponds to a movement to the left and reduces the discount rate. The two thin black lines from right to left hold $\eta$ fix at $\frac{2}{3}$ (D) and 2 (N), while increasing risk aversion. The two thin lines moving up hold RRA fix at 5 and 10, while increasing $\eta$. The cited estimates of the disentangled model all imply rates in the lowest corner of the shaded area between the thin black lines. It is $\mu_t = 2\%$ and $\sigma_t = 4\%$.

aversion in the disentangled model decreases the discount rate. The thin black lines going from right to left increase risk aversion while keeping $\eta = \frac{2}{3}$ (D) and $\eta = 2$ (N). The thin black lines moving up increase $\eta$ while fixing RRA at 5 and 10. All of the cited estimates of the disentangled model imply discount rates in the lowest corner of the shaded area between these lines. In contrast, an attempt to accommodate observed risk aversion of RRA $\in [5, 10]$ in the standard model would imply discount rates far above the 7% bound of the graph (plus pure time preference).

## 3.2 Stochastic Projects

The previous section derived an expression for the risk-free discount rate under intertemporal risk aversion. Here, the only uncertainty is about economic growth. Many long-term investment projects, however, are characterized by uncertain payoffs. A particularly important example is the evaluation of greenhouse gas mitigation and climate change adaptation projects. Once stochasticity of the project is introduced, the correlation between project payoff and uncertain economic growth becomes crucial for valuation. Lind (1982) argues for full positive correlation between project payoffs and economic baseline growth. Weitzman (2007) points out that this standard approach to cost benefit analysis does not apply to climate change projects. The major areas impacted by climate change would be " 'outdoor' aspects (broadly defined) like agriculture, coastal recreational areas, and natural landscapes", which are little correlated to technological progress. Moreover, some of these impacts directly affect utility rather than production. Various economists used related arguments to promote the use of the risk-free rate for the assessment of climate change projects. Indeed, the risk-free rate coincides with the discount rate for an uncorrelated stochastic project (see below).

A different correlation is implied by some common integrated assessment models. As pointed out by Nordhaus (2008), a high growth realization implies more production and, thus, more emissions. Then, damages and abatement payoff are both high. The resulting positive correlation between growth (or wealth) and project payoff is driven by the production-emissions-damage link. I add a third consideration driving correlation in the climate change context. The causal chain pointed out by Nordhaus (2008) and captured in his integrated assessment model DICE relies on the exogenous growth rate affecting emissions. However, if climate change turns out to have severe economic impact, then such impact is likely to affect the overall economic growth rate (Pindyck 2011). One transmission channel can be the mere straining of resources that would otherwise lead to technological progress, or by deviating technological progress into adaptation technology that merely serves to maintain the status quo. Dell, Jones & Olken (2008) find evidence that a similar channel indeed affects the growth rates in developing economies already at the moderate levels of climate change experienced in the past. Another transmission channel can be the major distributional change going along with fresh water scarcity, droughts, and agricultural yields in some regions of the world, which can trigger social tensions within a society as well as international conflicts. In general, all three correlation arguments (Weitman's, Nordhaus', and the one added here) apply to the evaluation of climate change related projects and the integrated assessment of climate change under uncertainty.

This section derives the discounting formula for projects that are correlated with economic baseline growth. I show that general risk attitude creates a much more important role for correlation than played in the standard model. For stochastic projects, the decision maker no longer trades a deterministic unit of consumption between the present and the future. Formally, she trades a marginal unit $dx_1$ of her current certain consumption $x_1$ against a marginal fraction $d\epsilon$ of a stochastic project $y$ with expected unit payoff, i.e. $\mathrm{E}y = 1$. The future project payoff $y$ is correlated with uncertain future baseline consumption $x_2$. The stochastic discount rate is characterized as $r = \ln \frac{d\epsilon}{-dx_1}$ for an intertemporal trade-off that leaves overall welfare constant:

$$0 = \frac{d}{dx_1} u(x_1) dx_1 + \beta \frac{d}{d\epsilon} \, f^{-1} \left[ \mathrm{E}_{p(x_2,y)} f \circ u(x_2 + \epsilon y) \right] \Big|_{\epsilon=0} d\epsilon \, . \tag{9}$$

I briefly comment on this extension of the social discounting model. First, for a certain project the marginal payoff $\epsilon y$ is certain and corresponds to $dx_2$ in the usual derivation of the risk-free social discount rate.[19] Second, marginality in the trade-off that defines the discount rate plays the same role as in any other economic price concept. The analytic formula for the discount rate will characterize (in rates) the present value willingness to pay for a marginal unit of the stochastic project. This willingness to pay depends on correlation. Third, I formalize a trade-off between a marginal current unit and the first marginal part of a finite stochastic unit project $y$.[20] Fourth, observe that the derivation does not rely on an optimal allocation of an adaptation-mitigation-portfolio – an assumption that would be inadequate in the climate change application I am particularly interested in.

I assume that $\ln y$ and the growth rate $g$ are jointly normally distributed with standard deviations $\sigma_y$, $\sigma_g$, and correlation $\kappa$. The expected growth rate is denoted $\mu_g$ and the condition $\mathrm{E}y = 1$ determines the remaining parameter of the distribution.[21]

**Proposition 2:** The stochastic social discount rate in the isoelastic setting with intertemporal risk aversion is

$$\begin{aligned} r = \delta \, &+ \eta\mu_g - \eta^2 \frac{\sigma_g^2}{2} - \mathrm{RIRA} \, \left|1 - \eta^2\right| \frac{\sigma_g^2}{2} \\ &+ \eta\kappa \, \sigma_g \sigma_y + |1-\eta| \, \mathrm{RIRA} \, \kappa \, \sigma_g \sigma_y \, . \end{aligned} \tag{10}$$

[19] In this case the formula above reduces to a more precise notation of what is commonly written as $\frac{d}{dx_2}$...$\mathrm{E}_p$...$u(x_2)$ – the difference being that the above notation makes explicit that (for $y = 1$) the decision maker trades a certain unit ($\epsilon$ or $dx_2$) while having an uncertain baseline $x_2$. Observe that also the first period derivative in equation (9) can be rewritten as $\frac{d}{d\epsilon_1} u(x_1 + \epsilon_1 y_1)|_{\epsilon_1=0, y_1=1} d\epsilon_1$.

[20] Modeling an infinitesimal share of a non-marginal unit project rather than a marginal project itself is important. It is well known that risk effects are second order effects. Therefore, stochasticity effects of an infinitesimal project would vanish.

[21] Let $\mu_y$ denote the expected value of (the marginal distribution of) $\ln y$. The condition $\mathrm{E}y = 1$ implies $\mu_y = -\frac{\sigma_y^2}{2}$. Making use of this constraint, it is $\mathrm{Var}(y) = e^{\sigma_y^2} - 1 \approx \sigma_y^2 + \frac{\sigma_y^4}{2}$. Thus, in the percentage range, $\sigma_y$ also approximates well the standard deviation of the project $y$ itself. I will refer to $\kappa$ as the correlation between the project and the baseline even though, more precisely, it is the correlation between $\ln y$ and the growth rate $g = \ln \frac{x_2}{x_1}$.

The second line distinguishes the stochastic social discount rate from its risk-free relative of the previous section. In the case of certainty about the project this second line vanishes ($\sigma_y = 0$). The same is true if the risk of the project and the baseline scenario are uncorrelated ($\kappa = 0$). The discount rate characterizes a marginal shift between current consumption and uncertain future consumption. Therefore, risk aversion with respect to the marginal project itself is a second order effect that does not find its way into the discount rate. Stochasticity of the small project only contributes through its interaction with baseline uncertainty. The second term in the second line of equation (10) distinguishes the correlation contribution in a model including intertemporal risk aversion from the correlation contribution in the standard model.

I assess the magnitude of the correlation contribution in the same growth scenario as before with $\mu = 2\%$ and $\sigma_y = \sigma_g = 4\%$. I discuss the additional assumption that the standard deviation of the project payoff equals that of growth uncertainty in the subsequent section, where I further elaborate the importance of the correlation term in the context of climate change evaluation. The standard correlation multiplier in Nordhaus' setting N is $\eta\sigma_g\sigma_y = 0.3\%$. In contrast, a disentangled assessment of the stochastic discount rate (where $\eta = \frac{2}{3}$ and RRA $= 9.5$) reduces the standard multiplier of the correlation coefficient to $\eta\sigma_g\sigma_y = 0.1\%$, but adds an intertemporal risk aversion multiplier of $|1-\eta|\,\text{RIRA}\,\sigma_g\sigma_y = 1.4\%$. The correlation contributions in the social discount rate are proportional to these multipliers and the correlation coefficient. For example, a correlation of $\kappa = \pm 0.5$ increases the social discount rate in the disentangled scenario by $\pm 0.8$ to an overall rate of 2.4% and 0.9%, respectively (keeping pure time preference at $\delta = 1.5\%$). These numbers make a strong point that under intertemporal risk aversion the correlation between the project payoff and economic growth is of first order importance for the discount rate. Figure 1 shows the dominant correlation multiplier caused by intertemporal risk aversion as a function of $\eta$ in light gray. It is the amount added (subtracted) from the risk free rate when accounting for full positive (negative) correlation and comprehensive risk attitude.

## 3.3 The Relevance of Future Risk

How relevant is uncertainty for the evaluation of long-term projects? Weitzman (2009) emphasizes the importance of uncertainty about climate sensitivity and economic damages for the assessment of climate change policies. His analysis builds crucially on generating fat tails in a standard expected utility model. His interesting findings have been criticized in a series of papers for their assumptions about the climate system as well as for stretching a too simple economic trade-off model beyond the domain where it is meaningful (Horowitz & Lange 2009, Pindyck 2009, Nordhaus 2009, Millner 2011). The current paper opens up a very different perspective on how uncertainty affects climate change evaluation. Even without uncertainty about the climate system itself, uncertainty about economic growth has a major impact on optimal climate policy. Including uncertainty about the climate system, the interaction and correlation between growth and project payoffs becomes a major ingredient for evaluating climate change and pricing carbon. The section analyzes the relevance of growth and project uncertainty in the model of comprehensive risk attitude.

The previous sections have shown that growth uncertainty reduces the discount rate. The two period model of those sections is equivalent to a simple iid growth model. However, once uncertainty becomes persistent it is well known that uncertainty not only changes the level of the discount rate, but also its term structure (Weitzman 1998, Azfar 1999). Making the time step explicit in equation (8) results in

$$r_T = \delta T + \eta \mu_T - \eta^2 \frac{\sigma_T^2}{2} - \text{RIRA} \left|1 - \eta^2\right| \frac{\sigma_T^2}{2} \,,$$

where variables indexed by $T$ depend on the time horizon (payoff period). For an iid process like a Brownian motion the variance grows linearly in futurity $T$. Then, if expected growth is constant ($\mu_T = T\mu$), payoffs in period $T$ are simply discounted at $T$ times the constant rate stated in equation (8). However, with persistent uncertainty the variance grows faster and the term structure of the discount rate falls, i.e. payoffs in the distant future are discounted at a relatively lower (yearly average) discount rate than payoffs in the close future.

In the following I analyze the importance of uncertainty for the evaluation of climate change in models with and without a comprehensive representation of risk attitude. The analysis relies on the primordial importance of the discount rate for climate change evaluation, impressively documented in Nordhaus's (2007) simulation discussed in the introduction. The reasoning does not rely on fat tails or diverging preference representations as in Weitzman (2009). I build the analysis around the following question: At what level of riskiness do uncertainty effects cancel the growth effect in the social discount rate? Growth discounting is the main economic driver of discounting. If uncertainty effects cancel the growth effect, then future costs and benefits are solely discounted with the pure rate of time preference $\delta$. I compare the necessary risk level between the standard and the disentangled model and analyze how this risk depends on the correlation between growth and project payoffs. A major advantage of approaching the uncertainty comparison in this way is that the uncertainty analysis is independent of pure time preference - a parameter whose magnitude is most contested in the debate.

The analysis uses a time horizon (or period) of 50 years. Instead of iid growth shocks, I now represent uncertainty in long-term growth (a highly intertemporally correlated event). I keep the assumption of an expected 2% yearly growth rate of consumption. In 50 years climate change is going to affect our planet severely under almost any forecast. It will affect economic activities directly as well as non-produced consumption. Some events like changes in precipitation patterns (or land loss) can also cause social unrest or war. Learning about climate sensitivity and, thus, an important ingredient into the severity of damages is predicted to take place on a similar time scale (Kelly & Kolstad 2001, Keller, Bolkerand & Bradford 2004).[22] I will measure uncertainty in terms of the variance of the growth process (and the project payoff) and translate it into the probability of being worse off tomorrow than today.

**Corollary 1**: The discount rate reduces to pure time preference, i.e. $r_T = \delta\, T$, if and only if,

1. in the case of the risk-free rate
$$\sigma_T = \left(\frac{1}{2}\Big(\eta + \frac{|1-\eta^2|}{\eta}\,\text{RIRA}\Big)\right)^{-\frac{1}{2}} \;\mu_T{}^{\frac{1}{2}}\;. \tag{11}$$
2. in the case of a risky project with $\sigma_T = \sigma_{g_T} = \sigma_{y_T}$
$$\sigma_T = \left(\frac{1}{2}\Big(\eta + \frac{|1-\eta^2|}{\eta}\,\text{RIRA}\Big) - \kappa\Big(1 + \frac{|1-\eta|}{\eta}\,\text{RIRA}\Big)\right)^{-\frac{1}{2}} \;\mu_T{}^{\frac{1}{2}}\;. \tag{12}$$
3. in the case of a general risky project
$$\sigma_{y_T} = \frac{(\eta^2 + |1-\eta^2|\,\text{RIRA})\,\sigma_{g_T} - 2\eta\frac{\mu_T}{\sigma_{g_T}}}{2\kappa\,(\eta + |1-\eta|\,\text{RIRA})}\;.$$

The conditions for eliminating the growth effect are identical for the risk-free rate and for the case of a risky project whose payoffs are uncorrelated to overall growth. More uncertainty is required if the risk terms are to cancel the growth term for a project whose

[22]While learning about the temperature effects of our emissions is supposed to take even longer than these 50 years, we are likely to learn more about economic growth in the meanwhile. The simple model assumes full serial correlation and, thereby, does not considering anticipated learning during the 50 year period under analysis. Yet the model is a good enough first order approximation for an assessment of magnitude and cuts straight to the point. A model incorporating anticipated learning would be significantly more complicated and had to include a multitude of additional assumptions.

Table 1 determines the risk that reduces the social discount rate to pure time preference.

| | | | | $\boldsymbol{\kappa = -1}$ | | $\boldsymbol{\kappa = -0.5}$ | | $\boldsymbol{\kappa = 0}$ | | $\boldsymbol{\kappa = 0.5}$ | | $\boldsymbol{\kappa = 1}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\eta$ | RRA | RIRA | $\sigma$ % | $\boldsymbol{p^\star}$ % | $\sigma$ % | $\boldsymbol{p^\star}$ % | $\sigma$ % | $\boldsymbol{p^\star}$ % | $\sigma$ % | $\boldsymbol{p^\star}$ % | $\sigma$ % | $\boldsymbol{p^\star}$ % |
| **N** | 2 | 2 | 0 | 71 | **7.9** | 82 | **11** | 100 | **16** | 140 | **24** | / | / |
| **D** | $\frac{2}{3}$ | 9.5 | 27 | 20 | **0.00002** | 23 | **0.0009** | 30 | **0.04** | 49 | **2** | / | / |
| S1 | 2 | 9.5 | 7.5 | 30 | 0.04 | 33 | 0.13 | 39 | 0.5 | 49 | 2 | 73 | 8.5 |
| S2 | 2 | 5 | 3 | 42 | 0.8 | 47 | 1.7 | 55 | 3.6 | 71 | 7.9 | 120 | 19 |

Notes: $\sigma = \sigma_y = \sigma_g$=standard deviation; $p^\star$=probability of being worse off in 50 years than today; $\kappa$=correlation coefficient between project and baseline risk. The $\kappa = 0$ case is equivalent to the risk-free social discount rate. Expected growth is a yearly 2% over 50 years. The settings are 'N' based on Nordhaus, 'D' for the disentangled parameter estimates, and sensitivity scenarios 'S1' and 'S2'.

payoffs are positively correlated to growth uncertainty. If the expected growth rate is simply $\mu_T = \mu\, T$ with a constant yearly expectation of $\mu$, then equations (11) and (12) show that only a standard deviation that evolves proportional to $\sqrt{T}$ leaves the yearly discount rate constant (at pure time preference). This fact illustrates once again that the term structure of the discount rate is flat only for iid uncertainty where $\sigma_T \propto \sqrt{T}$.

I analyze Corollary 1 using concrete probabilistic events. By $p^\star \equiv P(x_{50} \leq x_1)$ I denote the probability that anything including climate change causes society to be worse off in $T = 50$ years than today. It is the probability mass in the thin left tail of the growth distribution that implies a non-increasing standard of living between today and in 50 years. For the subsequent simulations, I keep expected consumption growth at a yearly rate of 2% and $T = 50$, which implies $\mu_T = 1$. Table 1 summarizes the numerical results for the different preference representations and for differing degrees of correlation. The table follows part 2 of Corollary 1 assuming $\sigma_{g_{T=50}} = \sigma_{y_{T=50}}$ (relaxed further below). In the intertemporally expected utility standard model with N preferences a standard deviation of unity eliminates growth discounting from the risk-free discount rate (equivalent to $\kappa = 0$). This standard deviation translates into the large probability of $p^\star = 16\%$ that society is equal or worse off in 50 years. In contrast, the disentangled approach with a comprehensive treatment of risk attitude implies $\sigma = 0.3$ and $p^\star = 0.04\%$. A chance of 4 in 10000 that we might not be better of in 50 years than today seems quite reasonable. Then, we should not discount the future for growth in the disentangled model. The probability necessary in the standard model is 400 times larger. The sensitivity scenario $S1$ in the table leaves relative Arrow Pratt risk aversion RRA at the estimate of 9.5, but increases aversion to intertemporal substitution $\eta$ to Nordhaus' value of 2. This change reduces intertemporal risk aversion, but still results in a probability $p^\star$ necessary to reduce discounting to pure time preference that is about two orders of magnitude smaller than in the N scenario with standard preferences. Sensitivity scenario $S2$ further reduces intertemporal risk aversion by also lowering the Arrow Pratt coefficient of risk aversion to 5. It still implies a $p^\star$ almost an order magnitude below that of the standard model.

The table also shows the important role played by the correlation between project payoff and growth uncertainty in the disentangled approach: correlation can change the probability $p^\star$ by several orders of magnitude. With disentangled preferences and a correlation coefficient $\kappa = -.5$ a probability of $p^\star = 0.0009\%$ is sufficient to make the risk terms cancel the growth effect, yielding a social discount rate that is equivalent to pure time preference. In contrast, with a correlation coefficient $\kappa = +.5$ a probability of $p^\star = 2\%$ would be needed. Under standard preferences these probabilities would be 8% and 24%, respectively. The stochasticity of the project with expected unit payoff can be characterized as follows. Let $p_y = P(y < 0.5 \vee y > 2)$ denote the probability that the project pays less than half or more than double of the expected unit. The interval $\sigma_y \in [0.2, 0.3]$ found for a non-positive correlation in the disentangled approach translates into $p_y \in [0.1\%, 2.2\%]$, whereas the corresponding interval $\sigma_y \in [.7, 1]$ in the N scenario translates into $p_y \in [35\%, 54\%]$. For perfect positive correlation $\kappa = 1$ the risk effects can only cancel the growth effect if the standard deviation of baseline growth exceeds that of the stochastic project. Thus, condition (12) has no solution.[23]

Disentangling the two different risks yields further insight. The left graph in Figure 3 depicts combinations of standard deviations that reduce the social discount rate to pure time preference. The right graph translates these standard deviations into the probabilities $p^\star$ that society is equal or worse off in 50 years under the expected yearly growth rate of 2% (growth uncertainty) and into the probability $p_y$ that the project pays out less than half or more than double the expected unit.[24] The dashed lines correspond to disentangled preferences (D), while the solid lines correspond to Nordhaus preferences (N). The graphs demonstrate that more uncertainty of the stochastic project decreases the baseline risk necessary for a reduction of the discount rate, if and only if, the correlation is negative. For a positive correlation, higher project uncertainty also requires a higher volatility of baseline growth if risk effects are to cancel the growth effect.

[23]The entries in Table 1 correspond to the intersections of the corresponding curves in Figure 3 with the dotted 45° line. The shape of the curves for $\kappa = 1$ demonstrates why there is no solution to equation (12) (no intersection of the $\kappa = 1$ curves with the dotted line).

[24]Note that such a translation into probabilities is possible because the marginal distribution of the bivariate normal only depends on the volatility in the remaining dimension. Note that the vertical range of the right graph corresponds to a $\sigma_y$-range of $[0, 0.5]$.

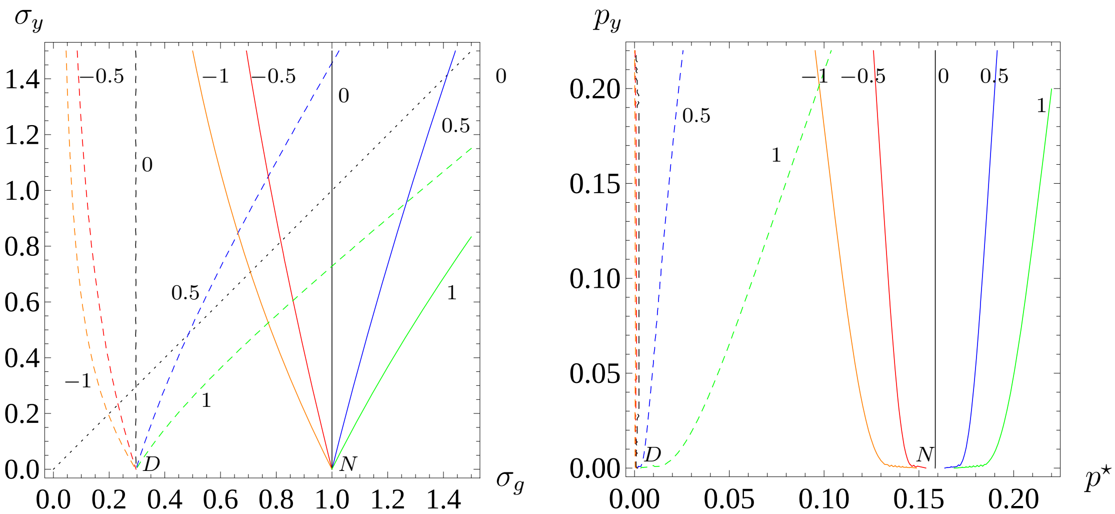


Figure 3 depicts the combinations of standard deviations (left) and probabilities (right) of baseline growth (horizontal axis) and project payoff (vertical axis) implying a discount rate reduction to the rate of pure time preference. $p^\star$ represents the probability of being worse off in 50 years than today under a normally distributed growth rate with expected value of 2% per year. $p_y$ represents the probability that the payoff of the stochastic unit project lies outside of the interval $[0.5, 2]$. The numbers labeling the curves denote the correlation $\kappa$ between baseline growth and project payoff. The dashed curves (originating at 'D') are based on the disentangled approach, the solid curves (originating at 'N') are based on Nordhaus' entangled preferences. The intersections of the curves in the left graph with the dotted line (identity) depicts the $\sigma$ values reported in Table 1.

The graphs clearly show the importance of the correlation coefficient, already at rather low levels of the project's payoff uncertainty $p_y$. Moreover, the graphs reiterate the order of magnitude difference resulting from entangled versus disentangled preferences.

# 4 Ambiguity Aversion and Second Order Uncertainty

## 4.1 Ambiguity

A different shortcoming of the intertemporally additive expected utility standard model dominating the social discounting debate and climate change assessment is its assumption that uncertainty can be described by a unique probability measure. In many real world applications these probability distributions or risks are unknown. Different strands of literature capture non-risk uncertainty under the names deep uncertainty, hard uncertainty, or ambiguity. In this paper, I employ and extend Klibanoff et al.'s (2005) model of smooth ambiguity aversion (KMM) to show how ambiguity affects the discount rate. In contrast to many models of ambiguity and deep uncertainty, the KMM model satisfies normative desiderata including time consistency and other rationality constraints. I show the close similarity of this ambiguity model to that of intertemporal risk aversion. The KMM model captures uncertainty about the correct objective probability distribution in terms of second order uncertainty: a subjective probability distribution over objective probability distributions or risk. The model is particularly interesting and applicable in the context of climate change and long-term economic growth: in both situations we face too little data for long term extrapolations and insufficient knowledge about the underlying model, which prevents a confident objective derivation of probabilities governing the future.

The basic structure of the model is similar to a Bayesian prior model. The Bayesian prior is interpreted as ambiguous second order uncertainty. The crucial distinction between the smooth ambiguity and the standard Bayesian model lies once more in the preference representation that accompanies the uncertainty model. In the standard expected utility model, a decision maker evaluates objective first order probabilities and subjective second order probabilities with the same degree of risk aversion; which, moreover, coincides with aversion to intertemporal substitution. In contrast, the KMM model incorporates the finding that individuals generally prefer objective risk to subjective risk. For this purpose, the model introduces a new measure of risk aversion for ambiguous lotteries (subjective second order probability distributions). I will explain that this measure of ambiguity is a close analogue to the measure of intertemporal risk aversion. The original KMM model keeps entangled attitudes to objective risk. By introducing ambiguity aversion, the model introduces intertemporal risk aversion only to subjective lotteries, while keeping intertemporal risk neutrality for objective lotteries. I extend the model to capture both, disentangled aversion to subjective and to objective risk. I show that the resulting social discounting model is a clone of the model discussed in the previous sections.

The decision-theoretic literature has developed a range of different approaches to capture situations of ambiguity. I briefly survey the most important ones in the remainder of this section. One way to characterize non-risk uncertainty is by extending the concept of probabilities to more general set functions called “capacities”. These set functions weigh possible events but are not necessarily additive in the union of disjoint events. Because of this non-additivity, the standard measure integral has to be exchanged for the more general Choquet integral in order to calculate expected utility, giving rise to the name “Choquet expected utility”. A second approach defines an evaluation functional that expresses beliefs in the form of sets of probability distributions rather than unique probability distributions. The first and simplest such representation goes back to Gilboa & Schmeidler (1989). Here a decision maker evaluates a scenario by taking expected values with respect to every probability distribution deemed possible and then identifies the scenario with the minimal expected value in this set.[25] A more general representation of this type is given by Ghirardato, Maccheroni & Marinacci (2004), Maccheroni, Marinacci & Rustichini (2006*a*), and, in an intertemporal framework, Maccheroni, Marinacci & Rustichini (2006*b*). There are several equivalence results between the Choquet approach and that of multiple priors as well as rank dependent utility theory where a decision maker uses distorted probabilities in an expected utility approach increasing the weights given to small probability events. In the climate change context, the main advantage of the smooth ambiguity model over these alternatives is its normative attractiveness achieved by maintaining time consistency and the essence of the independence axiom. Just as importantly for my purposes, I want to show that the KMM model is closely related to the model of intertemporal risk aversion and yields similar discounting results. Finally, its similarity to the Bayesian framework makes the model not only easy to interpret, but also allows me to relate to Weitzman's (2009) discourse on structural uncertainty.

## 4.2 The Generalized Model of Smooth Ambiguity Aversion

The section introduces the smooth ambiguity aversion model by Klibanoff et al. (2005) and, in an intertemporal setting, by Klibanoff et al. (2009). It represents ambiguity (non-risk uncertainty) as second order probability distributions, i.e. probabilities over probabilities. The model introduces a different attitude for evaluating second order uncertainty as compared to first order risk. Translated into the simple setting of this paper, the generally recursive evaluation of the future writes as

$$V(x_1, p, \mu) = u(x_1) + \beta \Phi^{-1} \left\{ \int_\Theta \Phi \left[ \mathrm{E}_{p_\theta(x_2)} u(x_2) \right] d\mu(\theta) \right\} .$$

For a given parameter $\theta$, the probability measure $p_\theta$ on the consumption space $X$ denotes first order or "objective" probabilities over consumption. The expectation operator takes expectations with respect to $p_\theta$. However, the parameter $\theta$ is unknown and so is the correct objective probability distribution. The probability measure $\mu$ denotes the prior

[25] Hansen & Sargent (2001) give conditions under which this approach is equivalent to what is known as robust control or model uncertainty, which again has overlapping representations with the model of constant absolute intertemporal risk aversion presented in Traeger (2007).

over the parameter $\theta \in \Theta$,[26] which translates into a prior over the right probability distribution $p_\theta$.

In Klibanoff et al.'s setting, the utility function $u$ corresponds to the utility function of the standard model. It jointly captures aversion to intertemporal substitutability and "objective" or first order risk. The function $\Phi$ captures additional aversion with respect to second order uncertainty, which they call *ambiguity aversion*. Note that, for $\Phi$ linear, the model collapses to the standard Bayesian model. The coefficient describing relative ambiguity aversion is defined as

$$\text{RAA} = \frac{\Phi''(z)}{\Phi'(z)} \, |z| \ .$$

In this paper, I combine Klibanoff et al.'s model of ambiguity aversion with my model of intertemporal risk aversion leading to a welfare representation of the form

$$V(x_1, p, \mu) = u(x_1) + \beta \Phi^{-1} \left\{ \int_\Theta \Phi \left[ f^{-1} \mathrm{E}_{p_\theta(x_2)} f \circ u(x_2) \right] d\mu(\theta) \right\} \ . \tag{13}$$

In this generalization, $u$ characterizes aversion to intertemporal substitution only, $f$ characterizes intertemporal risk aversion, and $\Phi$ characterizes ambiguity aversion.[27]

In the representation of equation (13), ambiguity aversion characterizes attitude with respect to second order uncertainty similar to the way that intertemporal risk aversion characterizes attitude with respect to first order risk. This parallel is a fundamental insight about the smooth ambiguity model and will also emerge in the expression for the discount rate. Moreover, the current generalized framework permits a threefold disentanglement of risk aversion, ambiguity aversion, and aversion to intertemporal substitution. The parallel between smooth ambiguity aversion and intertemporal risk aversion (and, thus, a reformulation of Epstein Zin preferences) helps to get a deeper understanding of Klibanoff et al.'s (2005) concept of smooth ambiguity aversion. To enable an analytic derivation of the social discount rate, I will once more revert to the isoelastic setting. In addition to the earlier assumptions of section 2.2 and equation (6)

[26] I pick a continuous parameter space $\Theta$, while this parameter space is finite in Klibanoff et al.'s (2009) axiomatization of the model. Note moreover that Klibanoff et al. (2005, 2009) setting features acts rather than probability measures on the outcome space.

[27] In an alternative representation, I could apply the inverse of the function $f$ characterizing intertemporal risk aversion in front of $\Phi^{-1}$ instead of its current position where it acts on the expected value operator. Then, the same preferences are represented with a different function $\Phi$ that would characterize only "access aversion" to ambiguity as opposed intertemporal risk aversion.

I assume $\Phi(z) = (\rho z)^{\varphi}$, which yields a coefficient of relative ambiguity aversion

$$\text{RAA} = \begin{cases} 1-\varphi \text{ if } \rho > 0 \\ \varphi - 1 \text{ if } \rho < 0\,. \end{cases}$$

## 4.3 The Social Discount Rate and Ambiguity about Growth

Weitzman (2009) recently argued that in the context of climate change the parameters of the distribution governing the growth process might not be known. Like Weitzman, I adopt a Bayesian setting to capture such a form of second order uncertainty. However, Weitzman sticks with the standard risk model underlying equation (2), in contrast, I introduce ambiguity attitude as formulated by Klibanoff et al. (2005, 2009) as well as intertemporal risk aversion. Taking the simplest example of Bayesian second order uncertainty, I assume that *expected* growth is itself a normally distributed parameter $\theta$ with expectation $\mu$ and variance $\tau^2$. Formally, that is $\text{E}(g|\theta) \sim N(\theta, \sigma^2)$ and $\theta \sim N(\mu, \tau^2)$, preserving the interpretation of $\mu$ as the overall expectation of the growth trend. The special case of Proposition 3 for RIRA = 0 and $\kappa = 0$ has independently been derived by Gierlinger & Gollier (2008).

**Proposition 3:** The stochastic social discount rate in the isoelastic setting with intertemporal risk aversion and ambiguity about expected growth is

$$\begin{aligned} r = \delta + \eta\mu_g - \eta^2\frac{\sigma_g^2+\tau^2}{2} &- \text{RIRA}\,\left|1-\eta^2\right|\frac{\sigma_g^2}{2} \\ &+ \eta\,\kappa\,\sigma_g\sigma_y + |1-\eta|\,\text{RIRA}\;\kappa\,\sigma_g\sigma_y \\ &- \text{RAA}\left|1-\eta^2\right|\frac{\tau^2}{2}\,. \end{aligned} \tag{14}$$

The first two terms on the right hand side reflect, once more, the discount rate in the standard Ramsey equation under certainty. The third term $-\eta^2\frac{\sigma^2+\tau^2}{2}$ reflects the well-known extension for risk. Note that the overall variance of the growth process is now $\sigma^2+\tau^2$ because of the additional layer of uncertainty characterized by the second order variance $\tau^2$. The second line gives the corrections if the project is stochastic. This correction remains as in the previous section. The third line characterizes the new contribution to intertemporal value development due to ambiguity aversion. The term is proportional to second order variance $\tau^2$, relative ambiguity aversion RAA, and the term $|1-\eta^2|$ already encountered in the correction of the social discount rate for intertemporal risk aversion. In fact, the contribution of ambiguity aversion is formally equivalent to the contribution of intertemporal risk aversion, replacing first by second order variance and RIRA by RAA. Proposition 3 provides a full disentanglement between the contributions already present under certainty, those arising under under risk but caused only by aversion to intertemporal fluctuations, the terms driven by intrinsic risk aversion, and those brought about by aversion to ambiguity.

Quantitatively, a decision maker who is more averse to ambiguity than to risk will lower the discount rate more for second order variance (ambiguity) than for first order variance (risk). Otherwise, the discussion from section 3 stays qualitatively the same. In general, an ambiguity averse decision maker will employ a lower (risk-free or stochastic) discount rate when the baseline scenario is ambiguous. He is willing to invest in a certain or stochastic project with relatively lower productivity than is a decision maker who is ambiguity neutral or just faces (first order) risk.

Relating my result to Weitzman (2009), I ignore everything but the first three terms on the right of equation (14). The only difference between these remaining terms and the standard equation (2) is the additional variance $\tau^2$ in the third term (standard risk term). This additional variance is a straightforward consequence of making the growth process more uncertain by introducing a prior (second order uncertainty) over some parameter of the growth process. In the case of the normal distributions adopted here, the variance simply adds up. From the given example, it is difficult to see how adding a Bayesian prior would bring the standard risk term back into the order of magnitude comparable to the other terms of the social discount rate. Instead of a doubling, a factor of $10 - 100$ is needed. The only way to reach this result is by sufficiently increasing the variance of the prior. Effectively, this is what Weitzman (2009) does in deriving what he calls a dismal theorem. He introduces a fat tailed (improper) prior whose moments do not exist. Consequently, the risk-free social discount rate in equation (14) goes to minus infinity implying an infinite willingness to transfer (certain) consumption into the future. Weitzman limits this willingness by the value of a (or society's) statistical life.[28] Instead of augmenting uncertainty, the above proposition introduces ambiguity aversion, i.e. the term $\text{RAA}\,|1-\eta^2|\,\frac{\tau^2}{2}$, into social discounting, reflecting experimental evidence that economic agents tend to be more afraid of unknown probabilities than they are of known probabilities (most famously, Ellsberg 1961).

Current estimates of the parameter RAA in the KMM model are significantly less reliable than in the intertemporal risk aversion framework and I refrain from a numerical analysis. Moreover, these models do not simultaneously estimate aversion to risk,

[28]Note that Weitzman (2009) puts the prior on the variance $\sigma$ rather than on the expected value of growth. He loosely relates the uncertainty to climate sensitivity. The above is a significantly simplified, but insightful, perspective on Weitzman's approach – abstracting from learning.

ambiguity, and intertemporal substitution.[29] However, the similarity of the ambiguity aversion effect to the direct effect of intertemporal risk aversion gives a good feeling for the magnitude by which a given degree of relative ambiguity aversion changes the social discount rate. Instead of redoing these simulations for ambiguity aversion, I will explore the effects of ambiguity about correlation between project payoffs and economic growth.

## 4.4 The Social Discount Rate and Uncertainty about Correlation

In section 3.2 I discussed different opinions on whether climate change related projects result in payoffs that are positively, negatively or not at all correlated to baseline risk. The standard approach in cost benefit analysis assumes full correlation of a project with the economic baseline risk. A similar correlation is supported by some integrated assessment models. Weitzman (2007) argues that major areas impacted by climate change are little correlated to technological progress as some of the impacts directly affect utility rather than production. He concludes that the correlation should be small. I made the point that climate change starts becoming a serious part of society's baseline risk. Moreover, mitigation and adaptation projects pay out most in states of the world where climate change turns out to be more serious. This reasoning introduces a negative correlation with part of the baseline risk.

In this subsection, I introduce uncertainty about correlation. Taking the opposite extreme of a perfectly known correlation, I assume an ignorant prior over the correlation coefficient, which permits an analytic solution. I am particularly interested in the difference between complete ignorance about the correlation and the assumption of an uncorrelated transfer. I assume that the correlation $\kappa$ between $\ln y$ and $g$ (see section 3.2) is uniformly distributed between $[-1, 1]$.

**Proposition 4:** The stochastic social discount rate in the isoelastic setting with intertemporal risk aversion and a uniform prior over correlation is

$$\begin{aligned} r = \delta + \eta\mu_g - \eta^2 \frac{\sigma_g^2}{2} - \text{RIRA } \left|1-\eta^2\right| \frac{\sigma_g^2}{2} \\ - \ln \left[ \frac{\sinh \{\eta \ \sigma_g \sigma_y + |1-\eta| \, \text{RIRA } \ \sigma_g \sigma_y\}}{\eta \ \sigma_g \sigma_y + |1-\eta| \, \text{RIRA } \ \sigma_g \sigma_y} \right] . \end{aligned}$$

[29]Paralleling this paper is a work by Ju & Miao (2009) using a similar model of threefold disentanglement. However, the authors fix Arrow-Pratt risk aversion exogenously to a level significantly lower than in the cited estimates of Vissing-Jørgensen & Attanasio (2003), Basal & Yaron (2004), and Basal et al. (2010), and then find an ambiguity measure in the range these papers estimate for standard risk aversion.

The terms in the first line resemble the risk-free social discount rate under intertemporal risk aversion derived in section 3. The second line captures the effect of uncertainty about the project and its correlation with the baseline growth. This additional component is of the form $h(z) = \ln\left[\frac{\sinh\{z\}}{z}\right]$, non-negative, and always reduces the discount rate as long as $z = (\eta + |1-\eta|\,\text{RIRA})\,\sigma_g\sigma_y \neq 0$. This latter condition is satisfied as long as the project and the baseline are stochastic and preferences do not simultaneously satisfy $\eta = 0$ and RIRA $= 0$. The function $h$ can be expanded into $h(z) = \frac{z^2}{6} - \frac{z^4}{180} + \frac{z^6}{2835} + O[z^7]$, where the first term already gives a good approximation for the magnitude relevant for the yearly discount rate. In the setting with yearly iid growth uncertainty in section 3.2 I found that $z$ was below one percent in all scenarios making $h(z)$ negligible. Note that the expression does not involve ambiguity aversion. While ambiguity aversion with respect to the baseline (section 4.2) is a first order effect, ambiguity with respect to the interaction of the project and the baseline becomes a second order effect not reflected in the social discount rate describing a marginal change.

In the 50 year scenario ignorance about correlation still only delivers a minimal deviation from the case of no correlation. I employ again the scenario introduced in section 3.3 and I assume a probability of $p^\star = 0.1\%$ that society will be worse off in 50 years than today. Then, in the disentangled scenario D ignorance over correlation would reduce the average discount rate from an uncorrelated 1.3% to 1.2%. In the first sensitivity scenario (S1), where RRA $= 9.5$ and $\eta = 2$, it would reduce the average discount rate from 2.7% to 2.6%. The differences in the second sensitivity scenario (average rate of 4.1%) and in Nordhaus's scenario (average rate of 5.1%) are negligible. The difference between the assumptions of ignorance over correlation and not being correlated grows as the risk increases. For $p^\star = 0.5\%$ ignorance as opposed to being uncorrelated reduces the average rate from 0.6% to 0.4% in the disentangled D scenario, and from 1.5% to 1.3% in the S1 scenario, still leaving the last digit unchanged in the S2 and the N scenarions (with average rates of 3.5% (S2) and 4.9% (N), respectively). Thus, the intertemporal evaluation of uncorrelated stochastic projects and projects with ignorance over the correlation coefficient are both well approximated by the risk-free social discount rate. Only a good estimate of the correlation will have a major impact on the evaluation.

# 5 Conclusions

Most long-term investment projects are subject to major uncertainties. The assessment of climate change is an important example. The recent discussion following the Stern review has put a spotlight on a particularly important aspect of intertemporal evaluation: the social discount rate. The discussion is framed almost exclusively in a standard intertemporally additive expected utility setting. I pointed out the limitations of this standard model and derived various contributions omitted in this framework. Three of these effects already arise in settings of pure risk. First, decoupling Arrow-Pratt risk aversion from intertemporal substitutability lowers the growth effect in the social discount rate. This is an immediate consequence of the empirical finding that the aversion to intertemporal consumption smoothing $\eta$ is overestimated when the parameter simultaneously has to capture the (generally stronger) aversion with respect to risk. Second, decoupling these two a priori independent preference parameters also removes an implicit assumption of (intertemporal) risk neutrality. I pointed out a simple characterization of intertemporal risk aversion and have shown that a term proportional to the coefficient of relative intertemporal risk aversion further reduces the risk-free social discount rate. The third contribution is for a stochastic project where payoffs are correlated to the economic baseline. Here, intertemporal risk aversion significantly increases the correlation effect in the social discount rate.

I derived conditions under which different risks and correlations imply that the social discount rate reduces to pure time preference, i.e. the risk terms cancel the growth effect. This risk is several orders of magnitude smaller when disentangling risk aversion from intertemporal substitution rather than employing standard preferences. The current literature argues for positive correlation between project payoffs and economic growth, as well as for the lack of correlation. I added an argument for negative correlation: when climate change becomes a major economic risk over the coming decades a bad state of the world decreases growth but increases the returns from adaptation or mitigation projects. Given the quantitative importance of correlation and aversion, I contrast the cases of no correlation with an evaluation where the decision maker is completely ignorant about the correlation between the project and economic baseline growth. While complete ignorance about correlation makes the social discount rate smaller than in the uncorrelated case, the magnitude of the effect is low. Overall, the discount rate for a stochastic project with ignorance about the correlation to economic growth can be well approximated by the risk-free rate. In general, I conclude that risk is of first order importance to social discounting in long-term cost benefit analysis and climate change assessment. This is true in the case of thin tailed probability distributions as soon as risk attitude is modeled comprehensively. Moreover, it is highly important to assess possible correlations carefully.

A further correction to the social discount rate stems from aversion to ambiguity.

Experimental evidence shows that decision makers are more averse to uncertainty in situations where uncertainty cannot be specified as risk. In the context of climate change, these situations of ambiguity (or hard uncertainty) are ubiquitous. I use the smooth ambiguity model to capture this distinction in uncertainty and in uncertainty attitude. Moreover, I merge this model with the model of intertemporal risk aversion. The resulting model permits a threefold disentanglement between risk attitude, consumption smoothing preference, and ambiguity attitude. I point out the similarities between ambiguity attitude and intertemporal risk aversion, in general, and derive that ambiguity aversion has an analogous influence on the social discount rate as does intertemporal risk aversion. The analytic derivations in this paper and their quantitative assessment serve as a rule of thumb how risk, correlation, and ambiguity change the social discount rate in the cost benefit analysis of climate related projects as well as other applications where time and uncertainty play an important role. Moreover, the modeling framework suggests itself for a less stylized numerical implementation in integrated assessment models.

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# Appendix A

**Proof of Proposition 1:** The first step of the proof calculates the marginal value of an additional certain unit of consumption in the second period ($dx_2$) in terms of first period consumption ($dx_1$). This value derives from the marginal trade-off that leaves welfare unchanged.

$$\begin{aligned}
&U(x_1,p) = \frac{x_1^\rho}{\rho} + \beta\frac{1}{\rho}\left[\mathrm{E}_p x_2^\alpha\right]^{\frac{\rho}{\alpha}} \\
\Rightarrow\ & dV(x_1,p) = x_1^{\rho-1}dx_1 + \beta\frac{1}{\alpha}\left[\mathrm{E}_p x_2^\alpha\right]^{\frac{\rho}{\alpha}-1}\ \mathrm{E}_p \alpha x_2^{\alpha-1}dx_2 \overset{!}{=} 0 \\
\Rightarrow\ & x_1^{\rho-1}dx_1 = -\beta\left[\mathrm{E}_p x_2^\alpha\right]^{\frac{\rho}{\alpha}-1}\ \mathrm{E}_p x_2^{\alpha-1}dx_2 \\
\Rightarrow\ & \frac{dx_1}{dx_2} = -\beta\left[\mathrm{E}_p\left(\frac{x_2}{x_1}\right)^\alpha\right]^{\frac{\rho}{\alpha}-1}\ \mathrm{E}_p\left(\frac{x_2}{x_1}\right)^{\alpha-1} \\
\Rightarrow\ & \frac{dx_1}{dx_2} = -\beta\left[\mathrm{E}_p e^{\alpha\ln\frac{x_2}{x_1}}\right]^{\frac{\rho}{\alpha}-1}\ \mathrm{E}_p e^{(\alpha-1)\ln\frac{x_2}{x_1}} \\
\Rightarrow\ & \frac{dx_1}{dx_2} = -\beta\left[e^{\alpha\mu+\alpha^2\frac{\sigma^2}{2}}\right]^{\frac{\rho}{\alpha}-1}\ e^{(\alpha-1)\mu+(1-\alpha)^2\frac{\sigma^2}{2}}
\end{aligned}$$

$$\Rightarrow \frac{dx_1}{dx_2} = -\beta e^{\rho\mu+\alpha\rho\frac{\sigma^2}{2}-\alpha\mu-\alpha^2\frac{\sigma^2}{2}} e^{(\alpha-1)\mu+(1-\alpha)^2\frac{\sigma^2}{2}}$$

$$\Rightarrow \frac{dx_1}{dx_2} = -\beta e^{(\rho-1)\mu+(\alpha\rho+1-2\alpha)\frac{\sigma^2}{2}}$$

$$\Rightarrow \frac{dx_1}{dx_2} = -\beta e^{(\rho-1)\mu+(\alpha\rho+1-2\alpha)\frac{\sigma^2}{2}} .$$

The second step translates the relation into rates by defining the social discount rate $r = -\ln \frac{dx_1}{-dx_2} \left(= -\ln \frac{dx_2}{-dx_1}|_{\bar{U}}\right)$, the rate of pure time preference $\delta = -\ln\beta$, and $\eta = 1-\rho \left(= \frac{1}{\sigma}\right)$. Further below, I make use of the relation $1 = \frac{1-\eta}{\rho}$.

$$\Rightarrow r = \delta + (1-\rho)\mu - (\alpha(\rho-1)+1-\alpha)\frac{\sigma^2}{2} \tag{A.1}$$

$$\Rightarrow r = \delta + \eta\mu - \eta^2\frac{\sigma^2}{2} + (\eta^2 + \alpha(\eta+1) - 1)\frac{\sigma^2}{2}$$

$$\Rightarrow r = \delta + \eta\mu - \eta^2\frac{\sigma^2}{2} + (\eta^2 + \frac{\alpha}{\rho}(1-\eta)(\eta+1) - 1)\frac{\sigma^2}{2}$$

$$\Rightarrow r = \delta + \eta\mu - \eta^2\frac{\sigma^2}{2} + (\eta^2 + \frac{\alpha}{\rho}(1-\eta^2) - 1)\frac{\sigma^2}{2}$$

$$\Rightarrow r = \delta + \eta\mu - \eta^2\frac{\sigma^2}{2} - (1 - \frac{\alpha}{\rho})(1-\eta^2)\frac{\sigma^2}{2}$$

$$\Rightarrow r = \delta + \eta\mu - \eta^2\frac{\sigma^2}{2} - \text{RIRA}\,\left|1-\eta^2\right|\frac{\sigma^2}{2} . \tag{A.2}$$

□

**Proof of Proposition 2:** For the isoelastic specification and with the definition

$$U_2(\epsilon) = f^{-1}\left[\mathrm{E}_{p(x_2,y)} f \circ u(x_2+\epsilon y)\right] = \frac{1}{\rho}\left[\mathrm{E}_{p(x_2,y)}(x_2+\epsilon y)^\alpha\right]^{\frac{\rho}{\alpha}}$$

equation (9) translates into

$$x_1^{\rho-1}\,dx_1 + \beta\,\frac{d}{d\epsilon}U_2(\epsilon)\bigg|_{\epsilon=0} d\epsilon \stackrel{!}{=} 0 \tag{A.3}$$

In order to calculate $\frac{d}{d\epsilon}U_2(\epsilon)\big|_{\epsilon=0} d\epsilon$ the following definition is useful.

$$V_\epsilon(a,b) = \mathrm{E}_{p(x_2,y)}(x_2+\epsilon y)^a y^b . \tag{A.4}$$

Then

$$\left.\frac{d}{d\epsilon}U_2(\epsilon)\right|_{\epsilon=0} = \left.\frac{1}{\alpha}V_\epsilon(\alpha,0)^{\frac{\rho}{\alpha}-1}\alpha V_\epsilon(\alpha-1,1)\right|_{\epsilon=0}$$

$$= V_0(\alpha,0)^{\frac{\rho}{\alpha}-1}V_0(\alpha-1,1) \tag{A.5}$$

where equality between the first and the second line follows from Lebesgue's dominated convergence theorem. Analogously to step 1 in the proof of Proposition 1, I calculate with $z = \ln y$

$$V_0(\alpha,0) = x_1^\alpha \,\mathrm{E}_{p(x_2,y)}\Big(\frac{x_2}{x_1}\Big)^\alpha = x_1^\alpha \,\mathrm{E}_{p(g,z)}e^{\alpha g}$$

$$= x_1^\alpha \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\alpha g}\frac{e^{-\frac{1}{2(1-\kappa^2)}\left[\left(\frac{g-\mu_g}{\sigma_g}\right)^2+\left(\frac{z-\mu_y}{\sigma_y}\right)^2-2\kappa\left(\frac{g-\mu_g}{\sigma_g}\right)\left(\frac{z-\mu_y}{\sigma_y}\right)\right]}}{2\pi\sigma_g\sigma_y\sqrt{1-\rho^2}}dg\ dz \tag{A.6}$$

$$= x_1^\alpha \, e^{\alpha\mu_g+\alpha^2\frac{\sigma_g^2}{2}} \,. \tag{A.7}$$

Similarly,

$$V_0(\alpha-1,1) = x_1^{\alpha-1}\,\mathrm{E}_{p(x,y)}\Big(\frac{x_2}{x_1}\Big)^{\alpha-1} y = x_1^{\alpha-1}\,\mathrm{E}_{p(g,z)}e^{(\alpha-1)g+z}$$

$$= x_1^{\alpha-1}\, e^{-(1-\alpha)\left[\mu_g-(1-\alpha)\frac{\sigma_g^2}{2}+\kappa\sigma_g\sigma_y\right]+\mu_y+\frac{\sigma_y^2}{2}}$$

so that

$$\left.\frac{d}{d\epsilon}U_2(\epsilon)\right|_{\epsilon=0} = x_1^{\rho-\alpha+\alpha-1}e^{(\alpha\mu_g+\alpha^2\frac{\sigma_g^2}{2})(\frac{\rho}{\alpha}-1)}e^{-(1-\alpha)\left[\mu_g-(1-\alpha)\frac{\sigma_g^2}{2}+\kappa\sigma_g\sigma_y\right]+\mu_y+\frac{\sigma_y^2}{2}}$$

$$= x_1^{\rho-1}e^{(\rho-1)\mu_g+[\alpha(\rho-1)+(1-\alpha)]\frac{\sigma_g^2}{2}-(1-\alpha)\kappa\sigma_g\sigma_y+\mu_y+\frac{\sigma_y^2}{2}} \,. \tag{A.8}$$

Substituting the result into equation (A.3) and solving for the discount rate yields

$$r = \ln\frac{d\epsilon}{-dx_1} = \delta + (1-\rho)\mu_g - [\alpha(\rho-1)+1-\alpha]\frac{\sigma^2}{2}$$

$$+(1-\alpha)\kappa\sigma_g\sigma_y - \big(\mu_y + \frac{\sigma_y^2}{2}\big) \,.$$

The first line corresponds to equation (A.1) and, thus, equation (A.2), yielding the risk-free discount rate under intertemporal risk aversion. Moreover, the random variable $y$

was assumed to yield an expected value (project payoff) of unity, which implies

$$\mathrm{E}_{p(x,y)} y = e^{\mu_y + \frac{\sigma_y^2}{2}} \stackrel{!}{=} 1 \qquad \Rightarrow \qquad \mu_y + \frac{\sigma_y^2}{2} = 0\,,$$

eliminating the last bracket. Finally, $1-\alpha$ has to be to be expressed in terms of $\eta$ (capturing the effects of the standard model) and RIRA (capturing the additional effects of intertemporal risk averison). I find for $\rho > 0$ that

$$1-\alpha = 1-(1-\eta)(1-\mathrm{RIRA}) = \eta + (1-\eta)\,\mathrm{RIRA}$$

and for $\rho < 0$ that

$$1-\alpha = 1-(1-\eta)(1+\mathrm{RIRA}) = \eta - (1-\eta)\,\mathrm{RIRA}\ .$$

In both cases this yields

$$1-\alpha = \eta + |1-\eta|\,\mathrm{RIRA}\,, \tag{A.9}$$

which gives rise to the form stated in the proposition. □

**Proof of Corollary 1:**
In case 1 of the risk-free discount rate, equation (8) translates $r_{50} = 50\delta$ into the condition $\eta\, 50\mu \stackrel{!}{=} \eta^2 \frac{\sigma^2}{2} + \mathrm{RIRA}\ |1-\eta^2|\, \frac{\sigma^2}{2}$, which results in the stated equation for $\sigma$. Similarly in case 2, equation (10), $\sigma = \sigma_g = \sigma_y$, and $\eta\, 50\mu_g \stackrel{!}{=} \eta^2 \frac{\sigma_g^2}{2} + \mathrm{RIRA}\ |1-\eta^2|\, \frac{\sigma_g^2}{2} - \eta\kappa\ \sigma_g\sigma_y - |1-\eta|\,\mathrm{RIRA}\,\kappa\ \sigma_g\sigma_y$ yield the result. Without the condition $\sigma = \sigma_g = \sigma_y$ the same reasoning gives statement 3 of the corollary. □

**Proof of Proposition 3:**
Define for the isoelastic specification

$$U_2^a(\epsilon) = \Phi^{-1}\left\{\int_\Theta \Phi\left[f^{-1}\mathrm{E}_{p_\theta(x_2,y)} f\circ u(x_2+\epsilon y)\right] d\mu(\theta)\right\}$$
$$= \frac{1}{\rho}\left\{\int_\Theta \left[\mathrm{E}_{p_\theta(x_2,y)}(x_2+\epsilon y)^\alpha\right]^{\frac{\rho}{\alpha}\varphi} d\mu(\theta)\right\}^{\frac{1}{\varphi}}\ .$$

I have to solve once more the equation

$$dV(x_1,p,\mu) = x_1^{\rho-1}\,dx_1 + \beta\,\frac{d}{d\epsilon}U_2^a(\epsilon)\bigg|_{\epsilon=0} d\epsilon \stackrel{!}{=} 0 \tag{A.10}$$

for $\ln\frac{d\epsilon}{-dx_1}$. Making use again of the definition

$$V_\epsilon(a,b) = \mathrm{E}_{p_\theta(x_2,y)}(x_2+\epsilon y)^a y^b\,,$$

where $\theta$ replaces $\mu_g$ in $p_{(x,y)}$ of equations (A.4) and (A.6), I find

$$\begin{aligned}
\left.\frac{d}{d\epsilon}U_2^a(\epsilon)\right|_{\epsilon=0} &= \frac{1}{\rho}\frac{1}{\varphi}\left\{\int_\Theta V_\epsilon(\alpha,0)^{\frac{\rho}{\alpha}\varphi}d\mu(\theta)\right\}^{\frac{1}{\varphi}-1} \\
&\quad \left.\left\{\int_\Theta \frac{\rho}{\alpha}\varphi V_\epsilon(\alpha,0)^{\frac{\rho}{\alpha}\varphi-1}\alpha V_\epsilon(\alpha-1,1)d\mu(\theta)\right\}\right|_{\epsilon=0} \\
&= \left\{\int_\Theta V_0(\alpha,0)^{\frac{\rho\varphi}{\alpha}}d\mu(\theta)\right\}^{\frac{1}{\varphi}-1} \\
&\quad \int_\Theta V_0(\alpha,0)^{\frac{\rho\varphi}{\alpha}-1}V_0(\alpha-1,1)d\mu(\theta)\,. \qquad \text{(A.11)}
\end{aligned}$$

With the help of equation (A.7), the $\{\cdot\}$ expression calculates to

$$\begin{aligned}
\int_\Theta x_1^{\alpha(\frac{\rho\varphi}{\alpha})}e^{(\alpha\theta+\alpha^2\frac{\sigma_g^2}{2})(\frac{\rho\varphi}{\alpha})}d\mu(\theta) &= x_1^{\rho\varphi}e^{\rho\varphi\alpha\frac{\sigma_g^2}{2}}\int_\Theta e^{\rho\varphi\theta}\frac{e^{-\frac{1}{2}\left(\frac{\theta-\mu_g}{\sigma_g}\right)^2}}{\sqrt{2\pi}\sigma_g}d\theta \\
&= x_1^{\rho\varphi}e^{\rho\varphi\alpha\frac{\sigma_g^2}{2}}e^{\rho\varphi\mu_g+\rho^2\varphi^2\frac{\tau_g^2}{2}}\,.
\end{aligned}$$

Acknowledging the equality of equations (A.5) and (A.8) and their similarity to the second integrand in equation (A.11) (for $\rho \leftrightarrow \rho\varphi$), this second integral becomes

$$\begin{aligned}
&\int_\Theta V_0(\alpha,0)^{\frac{\rho\varphi}{\alpha}-1}V_0(\alpha-1,1)d\mu(\theta) \\
&= \int_\Theta x_1^{\rho\varphi-1}e^{(\rho\varphi-1)\theta+[\alpha(\rho\varphi-1)+(1-\alpha)]\frac{\sigma_g^2}{2}-(1-\alpha)\kappa\sigma_g\sigma_y+\mu_y+\frac{\sigma_y^2}{2}}d\mu(\theta) \\
&= x_1^{\rho\varphi-1}e^{[\alpha(\rho\varphi-1)+(1-\alpha)]\frac{\sigma_g^2}{2}-(1-\alpha)\kappa\sigma_g\sigma_y+\mu_y+\frac{\sigma_y^2}{2}}\int_\Theta e^{(\rho\varphi-1)\theta}d\mu(\theta) \\
&= x_1^{\rho\varphi-1}e^{[\alpha(\rho\varphi-1)+(1-\alpha)]\frac{\sigma_g^2}{2}-(1-\alpha)\kappa\sigma_g\sigma_y+\mu_y+\frac{\sigma_y^2}{2}}e^{(\rho\varphi-1)\mu_g+(\rho\varphi-1)^2\frac{\tau_g^2}{2}}\,.
\end{aligned}$$

Substituting these results back into equation (A.11) delivers

$$\begin{aligned}
\left.\frac{d}{d\epsilon}U_2^a(\epsilon)\right|_{\epsilon=0} &= x_1^{\rho\varphi(\frac{1}{\varphi}-1)}e^{(\rho\varphi\alpha\frac{\sigma_g^2}{2})(\frac{1}{\varphi}-1)}e^{(\rho\varphi\mu_g+\rho^2\varphi^2\frac{\tau_g^2}{2})(\frac{1}{\varphi}-1)} \\
&\quad x_1^{\rho\varphi-1}e^{[\alpha(\rho\varphi-1)+(1-\alpha)]\frac{\sigma_g^2}{2}-(1-\alpha)\kappa\sigma_g\sigma_y+\mu_y+\frac{\sigma_y^2}{2}}e^{(\rho\varphi-1)\mu_g+(\rho\varphi-1)^2\frac{\tau_g^2}{2}} \\
&= x_1^{\rho-1}e^{[\alpha(\rho-1)+(1-\alpha)]\frac{\sigma_g^2}{2}+(\rho-1)\mu_g+[\rho\varphi(\rho-1)+1-\rho\varphi]\frac{\tau_g^2}{2}-(1-\alpha)\kappa\sigma_g\sigma_y+\mu_y+\frac{\sigma_y^2}{2}}\,.
\end{aligned}$$

Substituting this result into equation (A.10) and solving for $r = \ln \frac{d\epsilon}{-dx_1}$ yields analogously to the proof of Proposition 2 the discount rate

$$r = \delta + \eta\mu_g - \eta^2 \frac{\sigma_g^2}{2} - \text{RIRA} \left|1-\eta^2\right| \frac{\sigma_g^2}{2} + \ \eta\,\kappa\ \sigma_g\sigma_y$$
$$+ \left|1-\eta\right| \text{RIRA}\ \ \kappa\ \sigma_g\sigma_y - [1 - 2\rho\varphi + \rho^2\varphi]\frac{\tau_g^2}{2}\,.$$

The last term can be rearranged to the form

$$[1 - 2\rho\varphi + \rho^2\varphi]\frac{\tau_g^2}{2} = [(1-\varphi) + \varphi(1-\rho) - \varphi\rho(1-\rho)]\frac{\tau_g^2}{2}$$
$$= [(1-\varphi) + (1-\rho)^2 + (\varphi-1)(1-\rho)^2]\frac{\tau_g^2}{2}$$
$$= [\eta^2 + (1-\varphi)(1-\eta^2)]\frac{\tau_g^2}{2} = \eta^2\frac{\tau_g^2}{2} + \text{RAA}\left|1-\eta^2\right|\frac{\tau_g^2}{2}\,,$$

completing the proof. □

**Proof of Proposition 4:**

Up to equation (A.11) the proof is identical to that of Proposition 3. In the next step, in $V_0(\alpha, 0)^{\frac{\rho\varphi}{\alpha}}$ the ambiguity parameter $\theta$ replaces $\kappa$ instead of $\mu_g$. Thus the first integral in equation (A.11) becomes

$$\int_\Theta x_1^{\alpha(\frac{\rho\varphi}{\alpha})}\, e^{(\alpha\mu_g + \alpha^2\frac{\sigma_g^2}{2})(\frac{\rho\varphi}{\alpha})} d\mu(\theta) = x_1^{\rho\varphi} e^{\rho\varphi\mu_g + \rho\varphi\alpha\frac{\sigma_g^2}{2}} \int_{-1}^{1} \frac{1}{2} d\theta$$
$$= x_1^{\rho\varphi} e^{\rho\varphi\mu_g + \rho\varphi\alpha\frac{\sigma_g^2}{2}}\ .$$

For the integrand of the second integral in equation (A.11), I find

$$V_0(\alpha - 1, 1) = x_1^{\alpha-1} e^{(\alpha-1)\mu_g + (\alpha-1)^2\frac{\sigma_g^2}{2} + (\alpha-1)\theta\sigma_g\sigma_y + \mu_y + \frac{\sigma_y^2}{2}}$$

delivering the integral

$$\int_\Theta V_0(\alpha,0)^{\frac{\rho\varphi}{\alpha}-1} V_0(\alpha-1,1) d\mu(\theta)$$
$$= \int_\Theta x_1^{\rho\varphi-\alpha} e^{\rho\varphi\mu_g+\rho\varphi\alpha\frac{\sigma_g^2}{2}-\alpha\mu_g-\alpha^2\frac{\sigma_g^2}{2}}$$
$$x_1^{\alpha-1} e^{(\alpha-1)\mu_g+(\alpha-1)^2\frac{\sigma_g^2}{2}+(\alpha-1)\theta\sigma_g\sigma_y+\mu_y+\frac{\sigma_y^2}{2}} d\mu(\theta)$$

$$= x_1^{\rho\varphi-1} e^{(\rho\varphi-1)\mu_g+(\rho\varphi\alpha-2\alpha-1)\frac{\sigma_g^2}{2}+\mu_y+\frac{\sigma_y^2}{2}} \int_{-1}^{1} e^{(\alpha-1)\theta\sigma_g\sigma_y} \frac{1}{2} d\theta$$

$$= x_1^{\rho\varphi-1} e^{(\rho\varphi-1)\mu_g+(\rho\varphi\alpha-2\alpha-1)\frac{\sigma_g^2}{2}+\mu_y+\frac{\sigma_y^2}{2}} \quad \frac{\sinh[(\alpha-1)\sigma_g\sigma_y]}{(\alpha-1)\sigma_g\sigma_y}.$$

Substituting these results back into equation (A.11) returns the second period welfare change in $\epsilon$:

$$\frac{d}{d\epsilon}U_2^a(\epsilon)\bigg|_{\epsilon=0} = x_1^{\rho\varphi(\frac{1}{\varphi}-1)} e^{(\rho\varphi\mu_g+\rho\varphi\alpha\frac{\sigma_g^2}{2})(\frac{1}{\varphi}-1)}$$
$$x_1^{\rho\varphi-1} e^{(\rho\varphi-1)\mu_g+(\rho\varphi\alpha-2\alpha-1)\frac{\sigma_g^2}{2}+\mu_y+\frac{\sigma_y^2}{2}} \quad \frac{\sinh[(\alpha-1)\sigma_g\sigma_y]}{(\alpha-1)\sigma_g\sigma_y}$$
$$= x_1^{\rho-1} e^{(\rho-1)\mu_g+(\rho\alpha-2\alpha-1)\frac{\sigma_g^2}{2}+\mu_y+\frac{\sigma_y^2}{2}} \quad \frac{\sinh[(\alpha-1)\sigma_g\sigma_y]}{(\alpha-1)\sigma_g\sigma_y}.$$

Substituting this result into equation (A.10) and solving for $r = \ln \frac{d\epsilon}{-dx_1}$ yields analogously to the proof of Proposition 2 the discount rate

$$r = \delta + \eta\mu_g - \eta^2\frac{\sigma_g^2}{2} - \text{RIRA}\,\left|1-\eta^2\right|\frac{\sigma_g^2}{2} - \ln\left[\frac{\sinh[(\alpha-1)\sigma_g\sigma_y]}{(\alpha-1)\sigma_g\sigma_y}\right].$$

By symmetry of the hyperbolic sine, the sign of $(\alpha-1)$ can be flipped simultaneously in the numerator and the denominator. Using equation (A.9) to substitute for $(1-\alpha)$ then yields the result stated in the proposition. □

# Appendix B

The following proposition formalizes how intertemporal risk aversion defined in the sense of equation (4) translates into the curvature of the function $f$ in a preference representation of the form (3).[30]

**Proposition 5:** Let preferences over $X \times P$ be represented by equation (3) with a continuous function $u : X \to \mathbb{R}$ and a strictly increasing and continuous function $f : U \to \mathbb{R}$, where $U = u(X)$ and $\beta = 1$.
a) The corresponding decision maker is (weakly) intertemporal risk averse [loving], if and only if, the function $f$ is concave [convex].
b) The corresponding decision maker is intertemporal risk neutral, if and only if, there exist $a, b \in \mathbb{R}$ such that $f(z) = az + b$. An intertemporal risk neutral decision maker maximizes intertemporally additive expected utility (equation 1).

**Proof of Proposition 5:** a) *Sufficiency* of axiom (4): The premise of axiom (4) translates with $\beta = 1$ into the representation (3) as

$$\begin{aligned} (x^*, x^*) \quad &\sim \quad (x_1, x_2) \\ \Leftrightarrow u(x^*) + u(x^*) &= \quad u(x_1) + u(x_2) \\ \Leftrightarrow \quad u(x^*) \quad &= \frac{1}{2}u(x_1) + \frac{1}{2}u(x_2) \end{aligned} \tag{A.12}$$

Writing the implication of the axiom in terms of representation (3) yields

$$\begin{aligned} (x^*, x^*) \succ \quad & (x^*, \frac{1}{2}x_1 + \frac{1}{2}x_2) \\ \Leftrightarrow u(x^*) + \geq \; & f^{-1}\left(\frac{1}{2}f \circ u(x_1) + \frac{1}{2}f \circ u(x_2)\right) . \end{aligned} \tag{A.13}$$

Combining equations (A.12) and (A.13) returns

$$\frac{1}{2}u(x_1) + \frac{1}{2}u(x_2) \geq f^{-1}\left(\frac{1}{2}f \circ u(x_1) + \frac{1}{2}f \circ u(x_2)\right) , \tag{A.14}$$

which for an increasing [decreasing] version of $f$ is equivalent to

$$\Leftrightarrow f\left(\frac{1}{2}u(x_1) + \frac{1}{2}u(x_2)\right) > \; [<] \; \frac{1}{2}f \circ u(x_1) + \frac{1}{2}f \circ u(x_2) .$$

Defining $z_i = u(x_i)$, the equation becomes

$$\Leftrightarrow f\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right) \geq \; [\leq] \; \frac{1}{2}f(z_1) + \frac{1}{2}f(z_2) . \tag{A.15}$$

[30]Recasting the proposition for a strictly decreasing continuous function $f : U \to \mathbb{R}$ turns concavity in statement a) into convexity [and convexity into concavity]. Replacing the definition of intertemporal risk aversion by its strict version given in footnote 12 switches concavity to strict concavity in the statement.

Because preferences are assumed to be representable in the form (3), there exists a certainty equivalent $x^*$ to all lotteries $\frac{1}{2}x_1 + \frac{1}{2}x_2$ with $x_1, x_2 \in X$. Taking $x^*$ to be the certainty equivalent, the premise and, thus, equation (A.15) have to hold for all $z_1, z_2 \in u(X)$. Therefore, $f$ has to be concave [convex] on $U(x)$ (Hardy, Littlewood & Polya 1964, 75).

*Necessity* of axiom (4): The necessity is seen to hold by going backward through the proof of sufficiency above. Strict concavity [convexity] of $f$ with $f$ increasing [decreasing] implies that equation (A.15) and, thus, equation (A.14) have to hold for $z_1, z_2 \in u(X)$. The premise corresponding to (A.12) guarantees that equation (A.14) implies equation (A.13) which yields the implication in condition (4). Replacing $\succeq$ by $\preceq$ and $\geq$ by $\leq$ in the proof above implies that the decision maker is intertemporal risk averse, if and only if, $f$ is convex [for an increasing version of $f$ and concave for $f$ decreasing].

b) The decision maker is intertemporal risk neutral, if and only if, $f$ is concave and convex on $u(X)$, which is equivalent to $f$ being linear.[31] However, a linear function $f$ cancels out in representation (3) and makes it identical to the intertemporally additive expected utility standard representation (1).

[31] Alternatively use $\sim$ and $=$ instead of $\succeq$ and $\geq$ in part a) and use Aczél (1966, 46).