GROWTH, LONGEVITY AND PUBLIC POLICY

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Abstract

This paper studies the optimal long-run public intervention in a two-period OLG model where the probability of surviving the first period and the length of the second period can be influenced by distinct policies. While the optimal size of public intervention depends on the extra-productivity of public spendings in longevity, its optimal structure is determined by (1) differences in the productivity of each policy; (2) how growth would influence each longevity aspect under laissez-faire; (3) the dependence of each longevity aspect on past achievements. Given competing effects, the optimal intervention can hardly, under additive expected lifetime utility, be strongly unbalanced.

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1 Introduction

The second part of the 20th century being characterized, in industrialized countries, by the coexistence of a substantial lengthening of life and a strong economic growth, it does not come as a surprise that economists have paid a particular attention to the relationship between economic growth and longevity.

Besides the numerous econometric studies on the growth-longevity relationship, longevity has been also introduced recently in growth models, in order to study the various effects of the length of life on accumulation processes, and, hence, on economic expansion.[3]Whereas pioneer models examined the effects of mortality on accumulation and growth while taking survival conditions as exogenous (see Ehrlich and Lui, 1991; de la Croix and Licandro, 1999; Pecchenino and Utendorf, 1999; Zhang et al, 2001; Boucekkine et al, 2002), a more recent group of models postulated endogenous survival conditions in order to be able to analyze the reciproqual relationships between prosperity and longevity (see Chakraborty, 2004; Aisa and Pueyo, 2004; Bhattacharya and Qiao, 2005; Finlay, 2005; Pestieau et al, 2006).

Despite their differences, all those models aimed at emphasizing that, while survival conditions, by affecting the temporal horizon of individuals, influence their investment decisions (in physical and human capitals) - and thus growth -, factors affecting accumulation and growth also influence, via several distinct channels, survival conditions. Those channels may take the form of health-enhancing public expenditures (as in Chakraborty, 2004), or of health-enhancing private spendings (as in Pestieau et al, 2006), or of both (as in Bhattacharya and Qiao, 2005), private and public spendings completementing each other.

This paper shares with the existing literature the goal of examining the growth-longevity relationship: its aim is, in a first stage, to study, within a two-period OLG model, the impact of public policy on growth and longevity, and, in a second stage, to examine the definition of the optimal long-run public intervention in an economy where longevity is endogenous and can be affected by public policy. However, this study differs from the literature on three grounds.

Firstly, whereas existing models introduce longevity either under the form of a probability of survival from one period of life to another (as in Chakraborty, 2004), or under the form of a second period of life whose length is a fraction of unity (as in Bhattacharya and Qiao, 2005), the present paper aims at modeling both the proportion of survivors and the maximum length of life. Such a modeling is required to be able to describe the actual evolution of survival conditions, which, as shown by Figure 1 in the case of France, involves both a rise in the proportion of people reaching high ages for a fixed maximum age at death (i.e. the rectangularization process) and a rise of the maximum age at death (i.e. a shift of the survival curve to the right).[4] As demographers emphasized, the rectangularization process and the rise in the maximum length of life are two

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[3]On empirical studies, see Preston (1975, 1980), Anand and Ravallion (1993), Pritchett and Summers (1996), Easterlin (1999), and Lichtenberg (2002).

[4]On the measurement of the rectangularization phenomenon, see Wilmoth and Horiuchi (1999), and Vallin and Berlinguer (2002).

distinct phenomena, which did not occur at the same periods and with the same intensities, so that these may be influenced by different factors.[5] In particular, public intervention may affect those two aspects of longevity in different ways, so that the study of optimal policy requires to distinguish those phenomena.

Secondly, the present paper, contrary to most existing models where the intervention of the government takes a single form, introduces the distinction between two kinds of public policies: on the one hand, "pure survival" expenditures, which affect the proportion of a cohort who survives the first period (e.g. public programs promoting security on roads); on the other hand, "pure longevity" expenditures, affecting the length of life of the people surviving the first period (e.g. scientific research against ageing diseases).
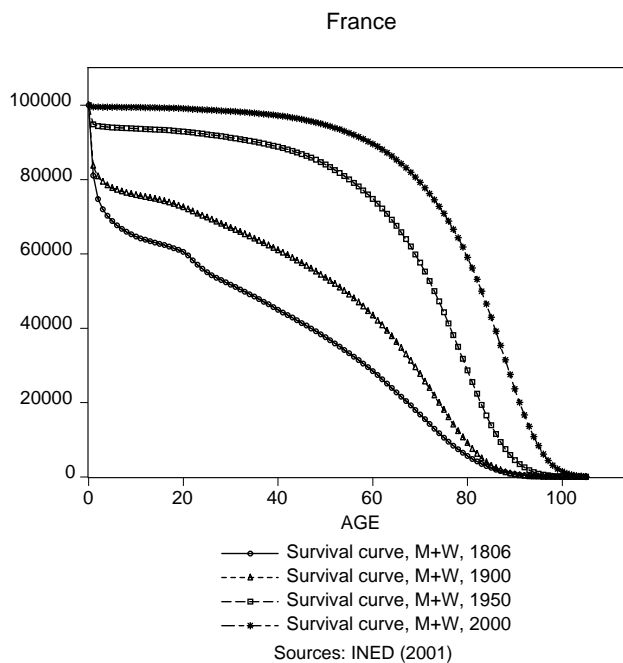
France

Sources: INED (2001)

Figure 1: Period survival curves, men and women, France.

Thirdly, while existing models assume no dependence of longevity achievements on the past, a dependence on the past is postulated in this paper, in the sense that a stock of common knowledge and customs on "how to live a healthy

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[5] For instance, Wilmoth and Horiuchi (1999) and Wilmoth et al (2000) emphasized that, in the case of Sweden, the rectangularization took place mostly between 1860 and the 1950s, whereas the maximum age at death started growing in a stronger way in the late 1960s. Regarding the various causes of rectangularization, see Nusselder and Mackenbach (1996).

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life", which affects longevity positively, is supposed to be accumulated and transmitted across generations.[6] The existence of some dependency - via customs - on past longevity achievements does not imply that mortality is not affected by "one-shoot" policies, which have to be implemented repeatedly. Hence, mortality is here supposed to be determined both by past achievements and by current policies.

In order to study the optimal size and structure of public intervention in an economy where the probability of survival from the first period to the second period and the length of the second period can be affected by distinct public policies, we shall, for analytical convenience, rely on a simplified model, where the production process only involves a unique generation of same-age individuals, unlike in human capital vintage models, such as Boucekkine et al (2002), where longevity is exogenous. Moreover, unlike in Pestieau et al (2006), there is no private investment in health decision: individuals cannot here affect their longevity.

This paper is organized as follows. Section 2 presents a two-period OLG model where the probability of surviving the first period is endogenous, while the length of the second period is fixed. This basic model is used to study the tax rate maximizing steady-state utility. Section 3 presents an extended framework, where the maximum length of life is endogenous. The optimal long-run allocation of governmental budget is then examined. Section 4 concludes.

2 The basic model

2.1 Environment

Let us consider a simple two-period OLG model with identical households within each generation. Each generation is denoted by its time of birth.

All individuals of a generation t live the first period for sure. During that period of unitary length, all individuals work, and save some money to fund their consumption during the second period, which is a period of retirement.

However, not all members of a cohort t will live the second period: only a proportion ϕ_{t+1}^t of generation t will do so ($0 \leq \phi_{t+1}^t \leq 1$).[7] Moreover, the survivors of generation t will enjoy a second period of length λ_{t+1}^t ($0 \leq \lambda_{t+1}^t \leq 1$).

To illustrate the distinction between ϕ_{t+1}^t and λ_{t+1}^t, Figure 2 shows the survival curves for particular values of these. If λ_{t+1}^t is equal to 0.5 (i.e. the maximum length of life is 1.5), and if ϕ_{t+1}^t is equal to 0.75, the area below the survival curve is the square of side 1 and the rectangular of length 0.5 and of height 0.75. However, if all individuals of generation t survive the first period (i.e. $\phi_{t+1}^t = 1$), the area below the survival curve becomes a rectangular (of

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[6] On the influence of customs on longevity, see Vallin et al (2002).

[7] Given that all members of a generation live the first period entirely, this amounts to assume that the risk of death is fully concentrated at the end of the period.

height 1 and of length 1.5). In that latter case, there is zero variance in the age at death.
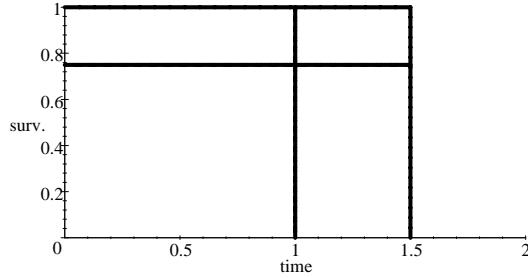

Figure 2: Survival curves.

Hence, while ϕ_{t+1}^t determines the heigth of the second rectangular, λ_{t+1}^t determines its length. The sum $1 + \phi_{t+1}^t \lambda_{t+1}^t$ is the well-known life expectancy (i.e. the average length of life), whereas $1 + \lambda_{t+1}^t$ constitutes what Vallin and Berlinguer (2002) call the "limit-longevity", that is, the maximum length of life that can be lived by a member of a cohort.[8]

2.2   Consumption and savings

Consumption during the first period, denoted by c_t, is equal to:

$$c_t = (1 - \tau)w_t - s_t \tag{1}$$

where w_t is the wage, τ is the tax rate, while s_t is saving. It is assumed that, at the end of the first period, each individual deposits his savings at a mutual fund. The mutual fund invests these savings in capital, and garantees a gross return equal to $\tilde{R}_{t+1}$ to the surviving old. The gross return $\tilde{R}_{t+1}$ is equal, under perfect competition, to R_{t+1}/ϕ_{t+1}^t (where R_{t+1} denotes the return on invested capital). For convenience, it is also supposed that a perfect annuities market exists.

Each individual chooses the savings s_t maximizing expected lifetime utility. Given that no one knows the actual proportion of survivors ϕ_{t+1}^t and the actual length of the second period λ_{t+1}^t, the saving decision is based on the longevity performances achieved by the previous cohort (i.e. ϕ_t^{t-1} and λ_t^{t-1}). Hence, under logarithmic utility, the problem is to choose s_t to maximize:

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[8] Although simplified - a survival curve is here reduced to 2 variables - the present model makes appear the dependence of life expectancy on the "limit-longevity".

$$\ln\left(w_t(1-\tau) - s_t\right) + \beta\lambda_t^{t-1}\phi_t^{t-1}\ln\left(\frac{1}{\lambda_t^{t-1}\phi_t^{t-1}}R_{t+1}s_t\right) \tag{2}$$

where β is a discount factor $(\beta > 0)$. The solution to this problem is:

$$s_t^* = \frac{\beta\lambda_t^{t-1}\phi_t^{t-1}}{1 + \beta\lambda_t^{t-1}\phi_t^{t-1}}(1-\tau)w_t \tag{3}$$

It appears here that the probability of survival and the length of the second period have a positive - but declining - influence on the propensity to save.

2.3 Production and capital accumulation

The production technology is specified as:

$$Y_t = AK_t^\alpha L_t^{1-\alpha} \tag{4}$$

where A is a productivity parameter $(A > 0)$, α is a parameter $(0 < \alpha < 1)$, while L_t is the labour force. In intensive terms, output per worker is:

$$y_t = \frac{Y_t}{L_t} = Ak_t^\alpha \tag{5}$$

The labour force L_t is supposed to grow at a constant rate n:[9]

$$L_{t+1} = L_t(1+n) \tag{6}$$

Profit maximization by firms implies:

$$\begin{aligned}
w_t &= AK_t^\alpha(1-\alpha)L_t^{-\alpha} = (1-\alpha)Ak_t^\alpha \tag{7}\\
R_t &= A\alpha K_t^{\alpha-1}L_t^{1-\alpha} = \alpha Ak_t^{\alpha-1} \tag{8}
\end{aligned}$$

It is supposed that there is a full depreciation of capital.
The capital market equilibrium $K_{t+1} = L_t s_t$ implies

$$k_{t+1} = \frac{(1-\alpha)(1-\tau)}{1+n}Ak_t^\alpha\frac{\beta\lambda_t^{t-1}\phi_t^{t-1}}{1 + \beta\lambda_t^{t-1}\phi_t^{t-1}} \tag{9}$$

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[9] As Alessandro Cigno kindly pointed out to me, that assumption is, given the varying longevity, quite strong. Extending this model to connect fertility to longevity is left to future research.

2.4 Longevity

2.4.1 The proportion of survivors

The proportion of survivors ϕ_{t+1}^t is supposed to be influenced by two factors.

On the one hand, it is affected by the global level of economic development. As it is widely acknowledged, richer countries are generally characterized by better survival conditions than poor countries.[10] However, as Dreze and Sen (1989) and Anand and Ravallion (1993) emphasized, countries with the same level of economic development are not equally good at converting wealth in health, and public policies constitute an important channel by which the wealth of Nations affects the health of Nations. Thus, we shall distinguish here two influences of wealth on ϕ_{t+1}^t: first, the influence of prosperity independently from public intervention, which is captured by the net wage $\tilde{w}_t$ (i.e. net of tax); second, the influence of the government, through public expenditures denoted by g_t.[11]

On the other hand, ϕ_{t+1}^t is also supposed to be affected by the accumulated stock of habits and common knowledge H_t on "how to live a healthy life", which can be regarded as a set of customs raising the probability to survive the first period.[12] That set of habits may explain why two countries with the same level of economic development and the same public policy do not necessarily exhibit identical survival conditions. For convenience, H_t is supposed to correspond to the proportion of survivors within the previous cohort (i.e. ϕ_t^{t-1}).

Hence, a simple production function for ϕ_{t+1}^t is given by:

$$\phi_{t+1}^t = 1 - \left[\frac{1 - B\phi_t^{t-1}}{1 + \gamma\tilde{w}_t + \delta g_t} \right] = \frac{B\phi_t^{t-1} + \gamma\tilde{w}_t + \delta g_t}{1 + \gamma\tilde{w}_t + \delta g_t} \tag{10}$$

where B is the habits parameter, while γ and δ reflect the impacts of $\tilde{w}_t$ and g_t on ϕ_{t+1}^t. In the rest of this paper, it is supposed that the influence of habits is, although non-negligible, weak (i.e. $0 < B < 1$), and that economic development has a globally positive influence on ϕ_{t+1}^t [i.e. $(1 - \tau)\gamma + \tau\delta > 0$, where τ is the tax rate], which seems to be, in the light of the empirical literature, a plausible assumption.

Expression (10) exhibits several plausible properties. First of all, the absolute impact of $\tilde{w}_t$ on ϕ_{t+1}^t is, provided $\gamma > 0$, positive, but declining with $\tilde{w}_t$ and g_t. The latter property is plausible, in the light of empirical evidence (see Dreze and Sen, 1989) suggesting that public policy can serve as a substitute for $\tilde{w}_t$ (unlike in Bhattacharya and Qiao, 2005, where public policy and private wealth are complementary). Moreover, the impact of g_t on ϕ_{t+1}^t is, under $\delta > 0$, positive, but declining with g_t and $\tilde{w}_t$. Finally, the influences of $\tilde{w}_t$ and g_t on

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[10]On this, see Pritchett and Summers (1996) for a panel data study.

[11]One should notice that the influence of economic development - independently of public intervention - on survival conditions is usually difficult to quantity. While prosperity seems to lead to lower mortality at the micro-level (see Valkonen, 2002), economic development at the macro-level may imply a deterioration of health (see Easterlin, 1999; Sartor, 2002).

[12]On the central influence of habits and customs on mortality, see Vallin et al (2002).

ϕ_{t+1}^t are here decreasing with ϕ_t^{t-1}, reflecting that the impact of economic development on survival conditions is likely to be lower when habits are stronger. On the contrary, the positive influence of habits on ϕ_{t+1}^t is falling with $\tilde{w}_t$ and g_t. This captures the intuition that it is at low $\tilde{w}_t$ and g_t that habits may have their largest impact on ϕ_{t+1}^t.

2.4.2 The length of the second period

The length λ_{t+1}^t of the second period of life enjoyed by survivors is supposed to be, throughout this Section, an exogenous constant λ ($\lambda > 0$):

$$\lambda_{t+1}^t = \lambda \tag{11}$$

This assumption constitutes a strong simplification: as demographers emphasized, the evolution of mortality over time has been such that not only has a larger proportion of people reached high ages, but the maximum age at death has also grown over time.[13] We shall thus endogenize λ_{t+1}^t in the next Section.

2.5 Government

The government confines itself to collecting a tax τ on wages, and uses the revenues from taxation to fund public policies g_t increasing the proportion of survivors of the first period ϕ_{t+1}^t. Those policies can take various forms. For instance, these can be public expenditures promoting security on roads.

We shall suppose that there is no possibility of public deficit, so that:

$$\tau w_t = g_t \tag{12}$$

2.6 Steady-states

Fixing $k_{t+1} = k_t$ in the capital accumulation equation (9) allows us to derive the kk locus, along which capital per worker is constant:

$$k = \left[\frac{A(1-\alpha)(1-\tau)\beta\lambda\phi}{(1+n)(1+\beta\lambda\phi)} \right]^{\frac{1}{1-\alpha}} \tag{13}$$

The first-order derivative of the kk locus with respect to ϕ is:

$$\frac{\partial k}{\partial \phi} = \frac{A(1-\tau)\beta\lambda \left[A(1-\alpha)(1-\tau)\beta\lambda \frac{\phi}{(1+n)(1+\beta\lambda\phi)} \right]^{\frac{\alpha}{1-\alpha}}}{(1+n)(1+\beta\lambda\phi)^2} > 0 \tag{14}$$

The second-order derivative of the kk locus with respect to ϕ is:

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[13] See Wilmoth et al (2000) and Vallin and Berlinguer (2002).

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$$\frac{\partial^2 k}{\partial \phi^2} = C\lambda \frac{\left[\frac{C\lambda\phi}{(1+n)(1+\beta\lambda\phi)}\right]^{\frac{2\alpha-1}{1-\alpha}} \frac{\alpha C\lambda(1+\beta\lambda\phi)}{(1+n)(1+\beta\lambda\phi)^2} - 2\lambda\left[\frac{C\lambda\phi}{(1+n)(1+\beta\lambda\phi)}\right]^{\frac{\alpha}{1-\alpha}}\beta(1-\alpha)}{(-1+\alpha)^2(1+n)(1+\beta\lambda\phi)^3} \tag{15}$$

where $C \equiv A(1-\alpha)(1-\tau)$.

In a similar way, fixing $\phi_{t+1}^t = \phi_t^{t-1}$ in expression (10) allows us to derive from this the $\phi\phi$ locus, along which ϕ is constant:

$$k = \left[\phi\frac{1-B}{A(1-\phi)(1-\alpha)(\gamma(1-\tau)+\delta\tau)}\right]^{\frac{1}{\alpha}} \tag{16}$$

The first-order and second order derivatives of the $\phi\phi$ locus are respectively:

$$\frac{\partial k}{\partial \phi} = \left[\phi\frac{1-B}{(1-\phi)E}\right]^{\frac{1-\alpha}{\alpha}} \frac{1-B}{\alpha(-1+\phi)^2 E} > 0 \tag{17}$$

$$\frac{\partial^2 k}{\partial \phi^2} = \frac{\left[\left[\frac{\phi(1-B)}{(1-\phi)E}\right]^{\frac{1-2\alpha}{\alpha}}\left[\frac{1-B}{(-1+\phi)^2 E}\right][(1-\phi)(1-\alpha)] + 2\left[\frac{\phi(1-B)}{(1-\phi)E}\right]^{\frac{1-\alpha}{\alpha}}\alpha\right]}{(B-1)^{-1}\alpha^2(-1+\phi)^3 E} \tag{18}$$

where $E \equiv A(1-\alpha)[\gamma(1-\tau)+\delta\tau]$. Under the plausible assumption of a globally positive impact of economic development on ϕ [i.e. $\gamma(1-\tau)+\delta\tau > 0$], those derivatives are strictly positive, so that the $\phi\phi$ locus is increasing and convex.
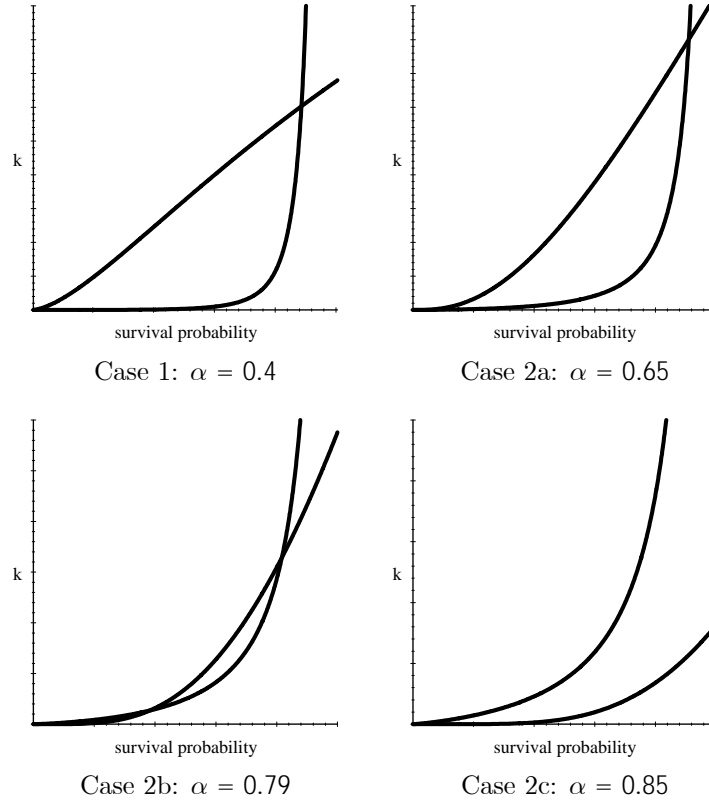
The issue of the existence of a steady-state can be formulated as the question of the existence of an intersection of the kk and $\phi\phi$ loci. It is straightforward to see, on the basis of the kk and $\phi\phi$ loci, that the two loci intersect at $(0,0)$. Regarding the existence of another, non-trivial equilibrium, two cases should be distinguished, depending on the value of α.

If $\alpha < 1/2$ (case 1), there must exist a unique non-trivial steady-state, provided the global influence of economic development on ϕ is positive (see Appendix A). Regarding the stability of that equilibrium, it is, under mild conditions, locally stable (see Appendix B).

If $\alpha \geq 1/2$, there may or may not exist a non-trivial equilibrium: as it is discussed in the Appendix, three cases should be distinguished: the existence of a unique non-trivial equilibrium (case 2a); the existence of two non-trivial equilibria (case 2b); the existence of no non-trivial equilibrium (case 2c).

Figure 3 illustrates those possible cases. On each graph, the $\phi\phi$ locus is the convex curve tending to $+\infty$ when ϕ tends to 1.

Figure 3: Equilibria in the simple model ($A = 20$, $\tau = 0.1$, $\beta = 0.99$, $\lambda = 0.8$, $n = 0$, $\gamma = 0.1$, $\delta = 0.3$ and $B = 0.5$).

Case 1: $\alpha = 0.4$

Case 2a: $\alpha = 0.65$

Case 2b: $\alpha = 0.79$

Case 2c: $\alpha = 0.85$

Regarding the stability of non-trivial equilibria, one should notice that, in cases 1 and 2a where there is a unique non-trivial steady-state, that equilibrium is likely to be locally stable.[14] To see this, it suffices to draw, on the phase diagram, the vertical and horizontal arrows on each area defined by the kk and $\phi\phi$ loci. Given that k tends to decrease when it is higher than the kk locus, and to increase below it, whereas ϕ tends to increase on the left of the $\phi\phi$ locus, and to decrease on the right of it, it is clear that, around the non-trivial equilibrium, all arrows are oriented towards this, suggesting that this is locally stable.

In case 2b, where the $\phi\phi$ and kk loci intersect twice (in addition to the trivial steady-state), it is easy to see, if one draws the arrows in the areas defined by the loci, that the intermediate equilibrium is not stable (while the high steady-state is). This observation is compatible with the existence of poverty traps: some level of development must be reached before being able to benefit from the positive dynamics and converge towards the high steady-state.

In case 2c, there exists no non-trivial equilibrium, and drawing the arrows suggests that, in that case, the unique equilibrium (i.e. the origin of axes) is stable. Such a catastrophic scenario can be explained by the fact that, when

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[14] See Appendix B for a formal study of stability.

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α is large, $1 - \alpha$ tends to zero, so that savings are low, which handicaps the accumulation of capital, whatever longevity is. However, this incapacity to accumulate capital implies that high survival probabilities can be neither reached nor sustained, so that both capital and survival can only converge towards zero.

2.7 Comparative statics

Let us now consider the effect, on the steady-state values of k and ϕ, of changing one parameter *ceteris paribus*. Those effects are summarized in Table 1.

Table 1: Comparative statics results

Parameter	Effect on k^*	Effect on ϕ^*
A	+	+
β	+	+
n	−	−
λ	+	+
B	+	+
γ	+	+
δ	+	+
τ	?	?

First of all, it is interesting to notice that, provided the global effect of prosperity on survival conditions is positive, the productivity parameter A and the discount factor β have a positive effect on both k^* and ϕ^*. Under the same - plausible - assumption, the cohort growth rate n has a negative effect not only on k^*, but, also, on ϕ^*. On the contrary, the length λ, by favouring savings, raises the steady-state values of k and ϕ.

The parameters of the survival production function - i.e. B, γ, δ - affect the steady-state ϕ positively. All this is hardly surprising: promoting the transmission of health-preserving customs and raising the productivity of public policies can only increase ϕ, so do attempts to make the development more "health-friendly" (i.e. raising γ). Moreover, given that ϕ affects savings positively, such changes also raise steady-state capital.

It is important to underline the unknown effect of the tax rate τ on steady-state k and ϕ. Whereas one may expect that raising τ reduces k^*, this is not necessarily the case, because a higher τ may raise k^* indirectly, *via* the positive effect of ϕ on the propensity to save. Moreover, raising τ does not necessarily raise ϕ^*: a higher τ may slowdown capital accumulation, which may lead to a lower ϕ^*. Hence, a numerical application is required to sign the effect of τ on the steady-state.

2.8 Numerical application: the optimal long-run tax rate

Whereas k and ϕ can, under some assumptions, grow "hand-in-hand", one should nonetheless keep in mind that a trade-off may exist between survival

conditions and capital accumulation. Raising ϕ^* is not costless, and may require a non-negligible sacrifice in terms of consumption. In order to illustrate the potential trade-off between steady-state survival and consumption, we shall proceed in three stages: we shall first study the tax rate maximizing steady-state survival, then, the tax rate maximizing steady-state capital per worker, and, finally, the tax rate maximizing average steady-state utility. Results are summarized in Table 2.

The tax rate maximizing steady-state ϕ, denoted by τ_ϕ^*, depends on the ratio γ/δ, that is, on the relative "productivity" of each determinant of survival conditions. This confirms what the intuition suggests: the more survival-enhancing economic development is "naturally", the smaller is the need for governmental intervention *ceteris paribus*. In the light of history, there can be no doubt that, although economic expansion may affect ϕ positively even without governmental intervention, targeted public programs are more productive, so that δ must exceed γ.[15] However, Table 2 suggests that, if one concentrates on long-run ϕ, this higher productivity does not suffice to justify a strictly positive τ_ϕ^*: public intervention requires a significant productivity gap. For instance, given our calibration, if δ is four times as high as γ, τ_ϕ^* is 33 %, but if δ is only twice as high as γ, then τ_ϕ^* is zero. In the latter case, the natural impact of economic development on ϕ is sufficiently high compared to the influence of g_t, so that *laissez-faire* is optimal.

Table 2 also illustrates that the tax rate maximizing steady-state capital per worker, denoted by τ_k^*, is lower than τ_ϕ^*. However, even if one neglects survival conditions, the optimal τ may still be strictly positive, for instance if economic development has a negative "natural" impact on ϕ, or if its positive impact on ϕ is too low compared to the one of public expenditures. This is explained by the fact that ϕ affects savings, so that, even if ϕ is not valued in itself, it may be worth intervening in the economy to guarantee a sufficiently high ϕ. Hence, the instrumental role of ϕ may suffice, in some circumstances, to justify public intervention. One should also notice that τ_k^* is, *ceteris paribus*, strictly decreasing with the productivity of public spendings.

The tax rate maximizing steady-state utility, denoted by τ_u^*, lies, not surprisingly, between τ_k^* and τ_ϕ^*. However, the gap between τ_k^* and τ_u^*, which exhibits a kind of inverted-U shape (as a function of γ), invites additional comments. First, one should notice that when γ tends to be negative, the gap between τ_k^* and τ_u^* tends to be, *ceteris paribus*, smaller. The reason for this result lies in the fact that, the lower γ is, the stronger is the need for intervening to sustain ϕ - even if one concentrates on capital only - so that the attention paid to both k and ϕ is stronger (although ϕ is then purely instrumental). Hence, the extent by which the goals of maximizing long-run capital and long-run utility differ is smaller when γ is low, so that, in that case, τ_k^* tends to converge towards τ_u^*. On the contrary, when γ becomes sufficiently high, the instrumental value of ϕ collapses, so that the goals of maximizing long-run utility and long-run capital differ now significantly, which is reflected by a large gap between τ_k^* and τ_u^*.

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[15] See Anand and Ravallion (1993).

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However, the latter statement does not imply that a higher γ increases necessarily the gap between τ_k^* and τ_u^*: once γ reaches some critical level with respect to δ, taxation becomes undesirable as far as steady-state utility is concerned, so that both τ_k^* and τ_u^* equal zero.

Table 2: Optimal steady-state tax rates τ^*($A = 20$, $\alpha = 0.5$, $\beta = 0.99$, $n = 0$).

λ	B	γ	δ	τ_ϕ^* %	τ_k^* %	τ_u^* %	λ	B	γ	δ	τ_ϕ^* %	τ_k^* %	τ_u^* %
.75	.5	−.05	.2	60	36	37	.75	.5	.075	.15	0	0	0
.75	.5	−.01	.2	52	23	24	.75	.5	.075	.175	12	0	0
.75	.5	0	.2	50	18	20	.75	.5	.075	.2	20	0	0
.75	.5	.01	.2	47	14	16	.75	.5	.075	.3	33	0	0
.75	.5	.025	.2	43	5	8	.75	.5	.075	.4	38	0	0
.75	.5	.035	.2	39	0	2	.75	.5	.075	.5	41	0	0
.75	.5	.05	.2	33	0	0	.75	.5	.075	.75	44	0	1
.75	.5	.075	.2	20	0	0	.75	.5	.075	.95	46	0	2
.75	.5	.085	.2	13	0	0	.75	.5	0	.3	50	15	17
.75	.5	.1	.2	0	0	0	.75	.5	−.01	.3	52	18	20
.25	.5	.075	.2	20	0	0	.75	.01	.075	.2	20	0	0
.5	.5	.075	.2	20	0	0	.75	.25	.075	.2	20	0	0
.75	.5	.075	.2	20	0	0	.75	.5	.075	.2	20	0	0
1	.5	.075	.2	20	0	0	.75	.99	.075	.2	20	0	0

The second part of Table 2 also illustrates that, whereas τ_ϕ^* is determined exclusively by the ratio γ/δ, this is not the case with τ_u^*, which depends - as τ_k^* - on the values of γ and δ, and not only on their ratio. Indeed, whereas the optimal tax rate is 20 % under γ equal to 0 and δ equal to 0.2, τ_u^* is equal to only 17 % once δ is equal to 0.3 (γ remaining fixed). Hence, from the long-run perspective, a rise in the productivity of the State implies necessarily a fall in the size of the State: once this is more productive, keeping its size constant would lead to a waste of resources and, in fine, a waste of utility. This suggests that τ_u^* is higher in countries where the State is less productive (e.g. in developing countries).

To summarize, this short discussion emphasizes that the optimal tax rate depends crucially on how "free" economic development and governmental intervention affect survival conditions. This conclusion does not constitute a surprise, but, in the light of the lack of consensus of empirical studies on the determinants of mortality, this conclusion highlights the difficulty to know whether actual public policies are optimal from a long-run perspective. It is important to stress that, for that purpose, knowing the extra-value added by public intervention does not suffice. The estimation of τ_u^* requires additional information on the "natural" influence of economic development, which remains hard to estimate.[16]

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[16] The natural influence of economic development on mortality is hard to quantify, because

3 Public policy with two instruments

Whereas Section 2 postulated that the maximum length of life $1 + \lambda_{t+1}^t$ is constant over time, this assumption constitutes, in the light of history, a strong simplification: during the last two centuries, there has been a strong rise not only in the proportion of people reaching high ages, but, also, in the maximum age at death.[17] Hence, although there has been, in the industrialized world, a strong rectangularization - i.e. ϕ_{t+1}^t grew more strongly than λ_{t+1}^t - there is a need for endogenizing λ_{t+1}^t. This task will allow us to examine the structure of the optimal public intervention, between policies raising the proportion of survivors ϕ_{t+1}^t and policies increasing the maximum longevity $1 + \lambda_{t+1}^t$.

3.1 Endogenous maximum length of life

We shall now suppose that λ_{t+1}^t is not constant, but can be affected either by the general prosperity, captured by the untaxed wage $\tilde{w}_t$, or by specific public policies (e.g. medical research against ageing diseases), denoted by r_t. Moreover, we shall suppose that λ_{t+1}^t can also be influenced by the accumulated stock of common knowledge and habits on "how to live a long life", approximated by the length of the second period enjoyed by the previous generation λ_t^{t-1}.[18] Hence, we shall postulate that λ_{t+1}^t is given by:

$$\lambda_{t+1}^t = 1 - \left[\frac{1 - R\lambda_t^{t-1}}{1 + \varepsilon\tilde{w}_t + \upsilon r_t} \right] = \frac{R\lambda_t^{t-1} + \varepsilon\tilde{w}_t + \upsilon r_t}{1 + \varepsilon\tilde{w}_t + \upsilon r_t} \tag{19}$$

where R is the habits parameter ($0 < R < 1$), while ε and υ reflect the impact of $\tilde{w}_t$ and r_t on λ_{t+1}^t. The model in Section 2 corresponds to the case where $R = 1$, whereas $\varepsilon = \upsilon = 0$. Here again, we shall suppose that economic expansion has a globally positive influence on λ_{t+1}^t, which can be written as $\varepsilon(1 - \tau) + \upsilon\tau(1 - \psi) > 0$, where ψ denotes the share of r_t in the public budget.

The length of the second period is, under $\varepsilon, \upsilon > 0$, affected positively by $\tilde{w}_t$ and r_t, but at a declining rate. Moreover, $\tilde{w}_t$ and r_t are competing channels by which economic development can affect the maximum length of life: the influence of each of these is declining with the level of the other. One should also notice that habits have a positive influence on λ_{t+1}^t, but declining with $\tilde{w}_t$ and r_t, which is plausible.

Whereas the modelling of λ_{t+1}^t looks like the one of ϕ_{t+1}^t, the two processes at work are likely to differ significantly. Actually, habits may play a stronger role for λ_{t+1}^t than for ϕ_{t+1}^t. The reason for this is that, among the main causes of death for young adults, death by car accident and death by suicide are causes for which there can be little dependency on past achievements, so that B is low (and probably lower than R). Public expenditures programs favouring security

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its impact on, among other things, the climate, the quality of air and water, are - although measurable - not easily identifiable as the cause of mortality changes (see Sartor, 2002).

[17] See Wilmoth et al (2000) and Vallin and Berlinguer (2002).

[18] On the influence of habits and customs on the mortality at high ages, see Vallin et al (2002).

on roads must be re-spent every year, otherwise the number of accidents tends to rise again. Hence, habits parameters B and R are likely to differ significantly.

3.2 Government

The government implements now two kinds of policies: on the one hand, public expenditures g_t affecting the proportion of survivors within a generation; on the other hand, "pure longevity" expenditures r_t, which raise the length of life enjoyed by survivors. Those expenditures can take various forms: medical research against ageing diseases, various preventive programs, etc.

The government's budget constraint is now:

$$\tau w_t = g_t + r_t \tag{20}$$

For convenience, we shall introduce the parameter ψ ($0 \leq \psi \leq 1$), which is the share of the budget dedicated to policies affecting ϕ_{t+1}^t, while the proportion $(1 - \psi)$ of the budget is dedicated to expenditures r_t.

3.3 Steady-states

Within this extended model, the capital accumulation equation is:

$$k_{t+1} = \frac{(1-\alpha)(1-\tau)}{1+n} A k_t^\alpha \frac{\beta \phi_t^{t-1} \lambda_t^{t-1}}{1 + \beta \phi_t^{t-1} \lambda_t^{t-1}} \tag{21}$$

The unique difference with respect to the basic model is that the length of the second period is no longer constant across generations.

As in the previous Section, fixing $k_{t+1} = k_t$ allows us to derive the kk locus:

$$k = \left(\frac{(1-\tau)(1-\alpha)A\beta\phi\lambda}{(1+n)(1+\beta\phi\lambda)} \right)^{\frac{1}{1-\alpha}} \tag{22}$$

The kk locus gives us, for each combination of ϕ and λ, the value of k such that k is constant. The length λ is no longer fixed here, so that the kk locus is not a curve in the (ϕ, k) space, but a plane in the (ϕ, k, λ) space.

The survival production function is now:

$$\phi_{t+1}^t = \frac{B\phi_t^{t-1} + \gamma(1-\tau)(1-\alpha)Ak_t^\alpha + \delta\tau\psi(1-\alpha)Ak_t^\alpha}{1 + \gamma(1-\tau)(1-\alpha)Ak_t^\alpha + \delta\tau\psi(1-\alpha)Ak_t^\alpha} \tag{23}$$

Fixing $\phi_{t+1}^t = \phi_t^{t-1}$ allows us to derive the $\phi\phi$ locus:

$$k = \left[\phi \frac{1-B}{A(1-\phi)(1-\alpha)(\gamma(1-\tau) + \delta\tau\psi)} \right]^{\frac{1}{\alpha}} \tag{24}$$

We shall, as in Section 2, make the plausible postulate that the global impact of economic development on ϕ_{t+1}^t is strictly positive.[19]

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[19] In the extended model, that assumption becomes $(\gamma(1-\tau) + \delta\tau\psi) > 0$.

Finally, fixing $\lambda_{t+1}^t = \lambda_t^{t-1}$ within the longevity production function:

$$\lambda_{t+1}^t = \frac{R\lambda_t^{t-1} + \varepsilon(1-\tau)(1-\alpha)Ak_t^\alpha + \upsilon\tau(1-\psi)(1-\alpha)Ak_t^\alpha}{1 + \varepsilon(1-\tau)(1-\alpha)Ak_t^\alpha + \upsilon\tau(1-\psi)(1-\alpha)Ak_t^\alpha} \qquad (25)$$
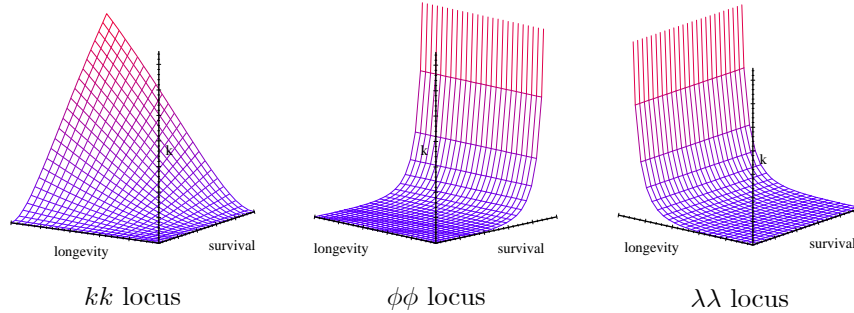
allows us to derive the $\lambda\lambda$ locus:

$$k = \left[\lambda\frac{1-R}{A(1-\lambda)(1-\alpha)(\varepsilon(1-\tau) + \upsilon\tau(1-\psi))}\right]^{\frac{1}{\alpha}} \qquad (26)$$

We shall assume that the global impact of economic development on λ_{t+1}^t is strictly positive.[20]

The question of the existence of an equilibrium, which could, in the basic model, be formulated as the search for an intersection of the kk and $\phi\phi$ loci in the 2-dimensional space (ϕ, k), must now be reformulated as the question of the intersection of the loci kk, $\phi\phi$ and $\lambda\lambda$ in the 3-dimensional space (ϕ, k, λ). In order to connect the present framework with the basic model, the three loci are, in Figure 5, represented as a collection of curves.

Actually, the basic model can be regarded as a particular case of the present one, where the selection of a specific value for λ amounts to "choose", from the kk plane shown in the first graph of Figure 5, a single curve among the ones constituting the kk locus. In a similar way, a single curve was selected from the $\phi\phi$ plane (see the second graph). Having selected those curves, corresponding to the kk and $\phi\phi$ loci for a specific λ, one could then discuss the possibility of their intersection.

Figure 5: The kk, $\phi\phi$ and $\lambda\lambda$ loci in the (ϕ, λ, k) space ($A = 20$, $\alpha = 0.5$, $\beta = 0.99$, $\tau = 0.25$, $\psi = 0.5$, $B = 0.5$, $\gamma = 0.1$, $\delta = 0.3$, $R = 0.5$, $\varepsilon = 0.1$, $\upsilon = 0.3$)
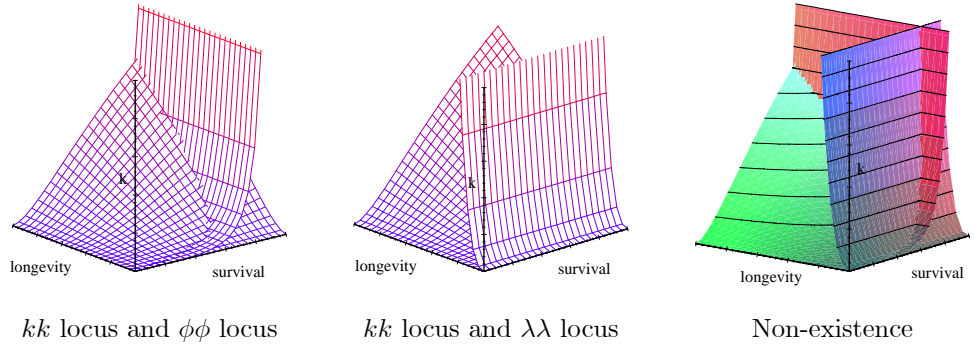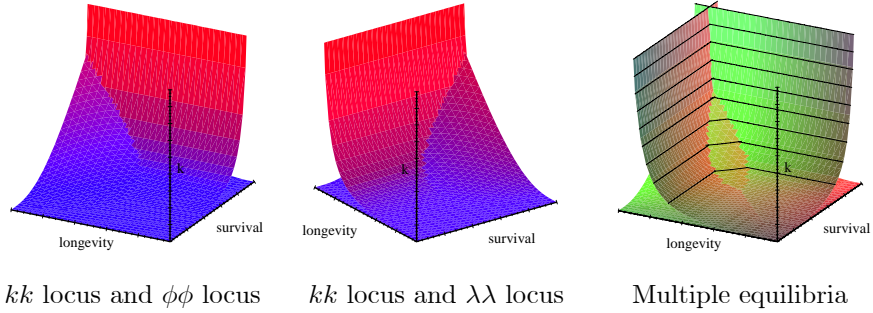

| kk locus | $\phi\phi$ locus | $\lambda\lambda$ locus |

In order to discuss the existence issue in the present framework, one strategy consists of deriving the equations describing the intersection of the kk locus with, on the one hand, the $\phi\phi$ locus, and, on the other hand, the $\lambda\lambda$ locus. Given that

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[20]That assumption is $(\varepsilon(1-\tau) + \upsilon\tau(1-\psi)) > 0$.

the first derived curve provides us, for different values of λ, the combinations of k and ϕ such that k and ϕ are constant, while the second curve provides us, for different values of ϕ, the combinations of k and λ that are constant, an equilibrium would lie at the intersection of those two curves, because an equilibrium point must, by definition, belong to the three loci. To illustrate this strategy, Figure 6 puts on the same graph the three loci from Figure 5. Those loci intersect not only at the origin, but also at another point, which is a non-trivial equilibrium.

Figure 6: Existence of an equilibrium ($A = 20$, $\alpha = 0.5$, $\beta = 0.99$, $\tau = 0.25$, $\psi = 0.5$, $B = 0.5$, $\gamma = 0.1$, $\delta = 0.3$, $R = 0.5$, $\varepsilon = 0.1$, $\upsilon = 0.3$)

longevity   survival
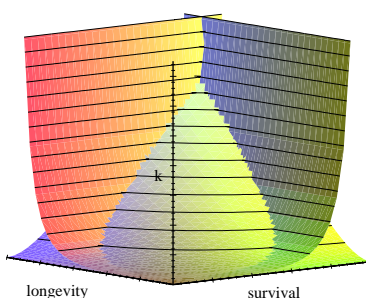
Regarding the conditions guaranteeing the existence of a non-trivial equilibrium, it is shown in the Appendix that, provided economic expansion has a globally positive influence on each aspect of longevity, the condition $\alpha < 1/3$ guarantees the existence of a non-trivial steady-state. Moreover, that condition guarantees also the unicity of the non-trivial equilibrium.

However, if that condition is not satisfied (i.e. $\alpha \geq 1/3$), neither the existence nor the unicity of a non-trivial steady-state is guaranteed: one may have no non-trivial steady-state, or, alternatively, several non-trivial equilibria may exist.

Let us first consider a case of non-existence. For that purpose, let us, for instance, suppose that α is 0.4, τ is 0.9, while ψ is 0.99. Whereas Section 2 suggests that, under that calibration, a non-trivial equilibrium must exist under an exogenous λ ($\lambda > 0$), this is no longer true once λ is endogenous. To see this, let us first present, on two distinct graphs, the intersections between, on the one hand, the kk and $\phi\phi$ loci, and, on the other hand, between the kk and $\lambda\lambda$ loci.

Figure 7: Non-existence of a non-trivial steady-state ($A = 20$, $\alpha = 0.4$, $\beta = 0.99$, $\tau = 0.9$, $\psi = 0.99$, $B = 0.5$, $\gamma = 0.1$, $\delta = 0.3$, $R = 0.5$, $\varepsilon = 0.1$, $\upsilon = 0.3$).

| kk locus and $\phi\phi$ locus | kk locus and $\lambda\lambda$ locus | Non-existence |

The left graph of Figure 7 illustrates clearly that, for a fixed, non-zero value of λ, there must exist a non-trivial equilibrium. Actually, as it was stressed above, fixing λ amounts to select a single curve from the myriad of curves constituting the kk locus, and, similarly, the single curve corresponding to the same value of λ from the $\phi\phi$ locus. It is straightforward to see that, provided the selected λ is strictly positive, the two selected curves - i.e. the kk and $\phi\phi$ loci from the previous Section - necessarily intersect at a non-trivial equilibrium. Nonetheless, there exists no non-trivial equilibrium in the extended model, because the curve defined by the intersection of the kk and $\lambda\lambda$ loci, which is illustrated on the second graph of Figure 7, only intersects the curve defined by the intersection of the kk and $\phi\phi$ loci at the origin of axes. Therefore, there is no non-trivial intersection of the three loci, so that no non-trivial equilibrium exists.

While the above numerical example is extreme, it suffices to illustrate that, once λ_{t+1}^{t} is endogenous, the existence of a non-trivial equilibrium is no longer guaranteed by the conditions stated in the basic model, so that stronger conditions are required. That stronger, sufficient condition is that $\alpha < 1/3$, so that a less capital-intensive production process is required to guarantee the existence of a non-trivial equilibrium. The intuition behind that stronger condition is the following: once λ_{t+1}^{t} is endogenous, the economy's take-off becomes more difficult, so that, in order to avoid stagnation, a less capital-intensive production process is required.

Regarding the unicity issue, it should be stressed that the extended model does not necessarily lead to a unique non-trivial equilibrium. Actually, the three loci may intersect at several points, as shown by Figure 8.

Figure 8: Multiple equilibria ($A = 20$, $\alpha = 0.7$, $\beta = 0.99$, $\tau = 0.25$, $\psi = 0.5$, $B = 0.5$, $\gamma = 0.1$, $\delta = 0.3$, $R = 0.5$, $\varepsilon = 0.1$, $\upsilon = 0.3$).

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| kk locus and $\phi\phi$ locus | kk locus and $\lambda\lambda$ locus | Multiple equilibria |

Having discussed the existence and unicity of an equilibrium, we can now discuss briefly its stability. Given that the stability issue was studied in depth in the basic model, we shall confine ourselves here to an intuitive discussion in the light of phase diagrams. Let us suppose that a non-trivial equilibrium exists, as on Figure 6. To see why that equilibrium is likely to be stable, one can draw, as in the 2-dimensional case, the horizontal and vertical arrows describing the dynamics of the economy outside the three loci. Given that the vertical arrows are directed upwards below the kk locus, and downwards above the kk locus, and given that the directions pointed to by the horizontal arrows associated to the $\phi\phi$ and $\lambda\lambda$ loci, the non-trivial equilibrium, if it exists, is likely to be locally stable. A similar procedure can be used to show that, under multiple non-trivial equilibria, the intermediate equilibrium is locally unstable. However, it should be reminded that such a diagram-based analysis is only an intuitive way to discuss the stability: in cases where the non-trivial equilibrium is close to the origin of axes, its stability, even local, depends also on the stability of zero, so that a formal analysis is required.

3.4   Comparative statics

Let us now consider the impact of various parameters on the steady-state. Table 3, which summarizes those effects, invites three remarks.

Firstly, the parameters A, β and n, which affect the accumulation of capital directly, also influence the steady-state levels of ϕ and λ in the same directions. This indirect influence follows from the postulated endogeneity of ϕ and λ: when an economy accumulates more capital, this has here a positive influence on ϕ [provided $\gamma(1-\tau) + \delta\psi\tau > 0$], and on λ [provided $\varepsilon(1-\tau) + \upsilon(1-\psi)\tau > 0$]. Therefore, any factor favouring capital accumulation also raises ϕ^* and λ^*.

Secondly, the parameters entering the survival production function - i.e. γ, δ and B - do not only affect ϕ^* positively, but, also, via their positive effect on capital accumulation, λ^*. For instance, if public expenditures promoting security on roads become more productive in terms of ϕ (i.e. δ is raised), this will also raise k^* and λ^*. This indirect influence comes from the savings decision: more security on roads increases the probability to live the second period, which makes individuals save more, which leads to a higher output, and, in fine, to a higher longevity.

Table 3: Comparative statics results

Parameter	Effect on k^*	Effect on ϕ^*	Effect on λ^*
A	+	+	+
β	+	+	+
n	−	−	−
B	+	+	+
γ	+	+	+
δ	+	+	+
R	+	+	+
ε	+	+	+
υ	+	+	+
τ	?	?	?
ψ	?	?	?

Thirdly, it should be stressed that, exactly as in the basic model, the influence of public policy parameters τ and ψ on the steady-state is unknown.

The unknown impact of τ follows from the fact that taxation has, as in the basic framework, two opposite effects on capital accumulation: on the one hand, a higher τ, by shrinking the share of the wage available for savings, tends to slowdown capital accumulation; on the other hand, if one supposes that governmental intervention brings some extra-value as far as survival conditions are concerned, a higher τ leads to a rise in ϕ and λ, which promotes capital accumulation. Hence, those opposite effects imply that the impact of raising τ on steady-state capital is unknown. But, given that ϕ and λ are also influenced by the untaxed wage, the effect of τ on their levels is also unknown: a higher τ does not necessarily lead to a higher longevity.

Finally, one should notice that the impact of ψ on the steady-state is unknown: in some cases, a higher proportion ψ of the government's budget allocated to survival expenditures g_t may affect the steady-state positively, whereas it may have, in other cases, a negative effect on not only k^* and λ^*, but, also, on ϕ^*. The reason why such a surprising result may occur is that changing ψ amounts to move the $\phi\phi$ and $\lambda\lambda$ loci in opposite directions (while the kk locus remains unchanged). Hence, given that the steady-state ϕ^* depends also on where the $\lambda\lambda$ locus lies, an optimal policy requires some kind of "balance" between the two kinds of public programs, balance whose precise form depends on the parameters determining ϕ and λ.

3.5 Numerical application: the optimal long-run policy

Let us now study, by means of a simple numerical application, the definition of the optimal allocation of the public budget between, on the one hand, spendings g_t, affecting the proportion of survivors ϕ_{t+1}^t, and, on the other hand, expenditures r_t, which raise the maximum length of life $1 + \lambda_{t+1}^t$. For that purpose (selecting the optimal ψ), we shall make two restrictions.

Firstly, we shall focus on the optimal long-run public policy, i.e., the policy maximizing the steady-state utility of a representative agent.[21] Such a focus on the steady-state is a simplification, because it ignores the welfare of all generations living before the steady-state is reached. However, given that the steady-state is, by definition, the state where the economy will remain forever once it has been reached, analyzing optimal public policy from that point of view is definitely relevant.[22]

Secondly, we shall examine the question of the optimal long-run ψ for a given fiscal capacity, that is, for a fixed τ. While this restriction is non-negligible, one should not exaggerate its strength: it is straightforward to imagine a situation where, because of political hysteresis, the fiscal pressure cannot be changed.

In order to study the optimal ψ, we shall proceed in two stages. In a first stage, we shall postulate that development is "neutral", in the sense that it affects ϕ_{t+1}^t and λ_{t+1}^t in an identical manner. Then, we shall consider the - more realistic - case where development is not neutral.

Under a neutral development, there are potentially two factors determining the optimal long-run ψ: on the one hand, the "productivities" of the two kinds of public expenditures g_t and r_t, and, on the other hand, the differences in the corresponding dependency on the past.

The crucial impact of the productivities on ψ^* is illustrated by Table 4, under the assumption of identical habits parameters ($B = R = 0.5$) and a neutral, but weakly positive, development ($\gamma = \varepsilon = 0.05$). When the productivities of the two public spendings are equal, then, given that the two production processes are exactly identical, the optimal ψ is 50 %.[23] Nevertheless, this case is purely theoretical: raising ϕ_{t+1}^t or λ_{t+1}^t is not equally difficult: the proportion of survivors of the first period is far easier to raise than maximum longevity.

As Table 4 illustrates, introducing some productivity gap does not leave ψ^* unchanged. Reducing υ tends, ceteris paribus, to lead to a higher ψ^*, suggesting that it is desirable to spend more where the productivity is higher. However, one should notice that, even when the impact of r_t on λ_{t+1}^t becomes as low as the "natural" impact of economic development, it is still desirable to dedicate a significant part of the budget to r_t. The intuition behind that result goes as follows: given the postulated utility function, which treats ϕ_{t+1}^t and λ_{t+1}^t in a symmetric way, some "balance" is always required between the resources dedicated to ϕ_{t+1}^t and to λ_{t+1}^t, because these are equally important for welfare.

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[21] Given that the postulated parameters do not guarantee the uniqueness of a steady-state, it is important to stress that ψ^* is here defined as the one maximizing the utility at the upper steady-state. Focussing on the upper steady-state amounts to suppose that this equilibrium can be reached given the initial conditions. However, the selected parameters always lead to an extremely low intermediate equilibrium, so that assuming that initial conditions are such that the poverty trap is avoided constitutes a weak postulate.

[22] One should also notice that focussing on the steady-state allows us to avoid the possibility of a difference between ex ante and ex post average lifetime utility (such a difference occurs outside the steady-state, because the actual values of λ_{t+1}^t and ϕ_{t+1}^t differ from their expected values outside the steady-state).

[23] This result follows from the fact that ϕ_{t+1}^t and λ_{t+1}^t are treated in the same way in the utility function of the representative agent, for whom life expectancy, equal to $1 + (\phi_{t+1}^t \lambda_{t+1}^t)$, is the unique thing that matters from a demographic point of view.

Table 4: Optimal ψ ($A = 20, \alpha = 0.5, \beta = 0.99, \tau = 0.25$)

γ	δ	B	ε	υ	R	$\psi^*(\%)$
0.05	0.3	0.5	0.05	0.3	0.5	50
0.05	0.3	0.5	0.05	0.2	0.5	52
0.05	0.3	0.5	0.05	0.1	0.5	65
0.05	0.3	0.5	0.05	0.075	0.5	74
0.05	0.3	0.5	0.05	0.05	0.5	91

Whereas Table 4 emphasized that differences in the productivity of public policies affect ψ^* under an equal dependence on the past for ϕ_{t+1}^t and λ_{t+1}^t, it should be stressed that the latter assumption is quite restrictive. Actually, as it was argued above, there exist good reasons to believe that the impact of the past is stronger in the determination of λ_{t+1}^t than in the determination of ϕ_{t+1}^t. For instance, the major causes of death for young adults - car accidents and suicides - can hardly be affected by past achievements, so that the influence of the past may, for ϕ_{t+1}^t, be lower than for λ_{t+1}^t.

As one may expect, differences in the degree of dependence on the past can hardly leave ψ^* unaffected. That intuition is confirmed by Table 5: once a differential in accumulation factors is introduced in favour of λ_{t+1}^t with respect to ϕ_{t+1}^t, ψ^* becomes significantly higher. It is, for instance, as high as 71 % under B equal to 0.5 and R equal to 0.6, against only 65 % under an identical dependence on the past (see Table 4). Not surprisingly, stronger differences in the degree of dependence on past achievements lead to higher ψ^*. But more surprising is that, under R equal to 90 %, the optimal policy is to spend exclusively on ϕ_{t+1}^t. This follows from the conjunction of the positive natural influence of development on λ_{t+1}^t with strong customs, which makes public intervention unnecessary from a long-run perspective.

While it was supposed so far that economic development is neutral, in the sense that it affects ϕ_{t+1}^t and λ_{t+1}^t in a similar manner, such a neutrality lacks empirical support. Actually, even though it is hard to separate the contributions of each - private and public - factor on longevity, the "natural" influence of economic development on ϕ_{t+1}^t is likely to have been significantly stronger than its influence on λ_{t+1}^t, as suggested by the historical fact that, at periods of limited public intervention, longevity progress began with a strong fall in mortality at low ages, before affecting mortality at all ages.[24]

As illustrated by Table 6, introducing an asymmetry in the natural impact of development on ϕ_{t+1}^t and λ_{t+1}^t is not without consequences on ψ^*.[25] If, for instance, one reduces ε from 0.05 to 0.04, everything else remaining unchanged with respect to Table 4, ψ^* falls from 65 % to 54 %. Reducing ε further tends to lead to even lower values of ψ^*. The tendency of ψ^* towards 0 when ε tends to 0 is not surprising: once the natural impact of prosperity on the maximum

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[24] See Vallin and Berlinguer (2002).
[25] For the convenience of the presentation, B and R are now equal as in Table 4.

length of life becomes very small, public intervention becomes the unique way to improve λ_{t+1}^t, so that the optimal budget is strongly unbalanced in favour of the aspect of longevity that cannot be improved without public intervention.

Table 5: Optimal ψ ($A = 20, \alpha = 0.5, \beta = 0.99, \tau = 0.25$)

γ	δ	B	ε	υ	R	ψ^*(%)
0.05	0.3	0.5	0.05	0.1	0.6	71
0.05	0.3	0.5	0.05	0.1	0.7	80
0.05	0.3	0.4	0.05	0.1	0.7	86
0.05	0.3	0.5	0.05	0.1	0.8	94
0.05	0.3	0.5	0.05	0.1	0.9	100

Hence, Table 6 suggests that, even though some public policies may be more efficient than others, what is crucial, at the end of the day, for the determination of ψ^*, is the precise extent by which each longevity aspect can grow without governmental intervention. If one has good reasons to believe that the length of the second period λ_{t+1}^t is likely to be less easily raised, without public intervention, than the proportion of survivors ϕ_{t+1}^t, this justifies, from a long-run perspective, a larger share of the budget to be spent on policies raising λ_{t+1}^t, even if the productivity of such policies is lower. Thus, all this tends to cast some doubts on the ψ^* derived under neutral economic development (Table 4).

Table 6: Optimal ψ ($A = 20, \alpha = 0.5, \beta = 0.99, \tau = 0.25$)

γ	δ	B	ε	υ	R	ψ^*(%)
0.05	0.3	0.5	0.04	0.1	0.5	54
0.05	0.3	0.5	0.03	0.1	0.5	43
0.05	0.3	0.5	0.02	0.1	0.5	32
0.05	0.3	0.5	0.01	0.1	0.5	22
0.05	0.3	0.5	0.005	0.1	0.5	17

Having studied the impact of a non-neutral development, we can now re-introduce differences in past-dependency, in order to examine the optimal policy once the major differences between the production of ϕ_{t+1}^t and λ_{t+1}^t are taken into account.

As Table 7 confirms, introducing differences in the degree of dependence on past longevity achievements affects ψ^* significantly. For instance, under ε equal to 0.03, a rise in R from 0.5 to 0.7 raises ψ^* from 43 % à 54 %. Moreover, under

ε equal to 0.01, raising R from 0.5 to 0.8 makes ψ^* grow from 22 % to 29 %. However, Table 7 suggests that the mere existence of a stronger dependence on the past for λ_{t+1}^t does not suffice, in itself, to justify an "unbalanced" budget structure favouring expenditures of kind g_t over expenditures of type r_t.

Table 7: Optimal ψ ($A = 20, \alpha = 0.5, \beta = 0.99, \tau = 0.25$)

γ	δ	B	ε	υ	R	ψ^*(%)
0.05	0.3	0.5	0.03	0.1	0.7	54
0.05	0.3	0.5	0.01	0.1	0.7	29
0.05	0.3	0.5	0.01	0.1	0.8	36
0.05	0.3	0.3	0.01	0.1	0.8	43
0.05	0.3	0.1	0.01	0.1	0.8	47

As shown by the last line of Table 7, even if one postulates that the past-dependency is, for λ_{t+1}^t, 8 times as strong as for ϕ_{t+1}^t, the optimal allocation of public budget is still close to 50-50. Although this result is obtained under a significant difference in the natural impact of prosperity on ϕ_{t+1}^t and λ_{t+1}^t, smaller differences could not, however, suffice to justify extremely unbalanced budget structures.

Hence, this numerical exercise tends to question the optimality, from a long-run perspective, of strongly unbalanced public budgets favouring expenditures oriented towards the proportion of survivors over expenditures enhancing the maximum length of life. If one concentrates on the steady-state, it is hard to see how an extremely unbalanced budget could be justified. Given that, under the postulated additive expected lifetime utility, ϕ_{t+1}^t and λ_{t+1}^t matter equally for the welfare of the representative agent, the competing effects at work suggest that the optimal budget structure should be, at least to some extent, balanced.

4   Conclusions

The aim of this paper was to examine the definition of the optimal long-run public intervention in a two-period OLG model where the proportion of survivors of the first period and the length of the second period can be affected by distinct public policies. For that purpose, we proceeded in two stages.

The presentation of the basic model - where the maximum length of life is fixed - allowed us to discuss the question of the optimal size of public intervention. As one may expect, the model suggests that, from a long-run perspective, there exists an optimal size of the State, which depends on the extra-value brought by public expenditures as far as the improvement of survival conditions is concerned. It may be the case that the natural impact of economic expansion

on survival conditions is so positive that laissez-faire is optimal. However, if development affects survival conditions negatively, a much larger public intervention is justified.

The extended model, where the maximum length of life is endogenous, allowed us to examine the optimal structure of public intervention. The optimal allocation of public resources - for a fixed fiscal pressure - was shown to depend on the productivities of the different policies, on the "natural" impact of economic development on these, and, also, on the degree of dependence on past achievements. It was argued that those different determinants play in opposite directions, so that, given the equal importance, in welfare terms, of the proportion of survivors and the maximum longevity, the optimal public budget should be relatively "balanced" between survival-enhancing policies and programs raising the maximum age at death. Actually, whereas the latter kind of spendings may be less productive than the former, and habits mechanisms probably stronger for the maximum length of life, those two considerations, which support a larger budget share dedicated to survival-enhancing expenditures, are - at least partly - counter-balanced by the fact that the natural influence of economic expansion on mortality at lower ages is generally larger than on mortality at higher ages, which justifies some balance in the budget.

Finally, it cannot be overemphasized here that this paper is only a first step in the study of the optimal public policy under endogenous longevity. Actually, the present study focussed on the long-run, and, thus, excluded all generations living before the steady-state is reached. Given that the well-being of people living in the transition is also worth being taken into account, and given that the transition may be quite long, it would be interesting to include generations living during the transition, and to discuss the sensitivity of the optimal public intervention to the postulated concept of intergenerational justice.

Moreover, while this study emphasized that the optimal long-run policy depends on how economic development affects mortality, it should be stressed that other factors may also influence the optimal public intervention. Among other things, the precise form of individual preferences may also play a crucial role. Whereas we postulated a utility function where ϕ and λ are treated in a symmetric way, such an assumption may be unrealistic: individuals may exhibit what Bommier (2003) calls "net" risk-aversion with respect to the length of life. Such an attitude, by implying a different weighting of ϕ and λ, may affect the structure of the optimal intervention. The optimal policy is also likely to be significantly sensitive to the postulated social welfare function (which was here the standard average view), and, especially, to the associated concerns for intragenerational inequality and numbers. Another factor that may influence the optimal public intervention consists of institutions, such as pensions schemes (see Pestieau et al, 2006). Hence, much work remains to be done in the future.

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5 References

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6 Appendix

6.1 Appendix A: Existence of steady-states in the basic model

Let us first rewrite the $\phi\phi$ locus as:

$$\phi = \frac{Ek^{\alpha}}{Ek^{\alpha} + (1-B)} \tag{27}$$

where $E \equiv A(1-\alpha)((1-\tau)\gamma + \tau\delta)$. Given our assumptions ($A > 0$, $0 < \alpha < 1$ and $(1-\tau)\gamma + \tau\delta > 0$), E is strictly positive.

Substituting for k while using the kk locus allows us to rewrite the above expression as:

$$\phi = \frac{E\left[\frac{C\beta\lambda\phi}{(1+n)(1+\beta\lambda\phi)}\right]^{\frac{\alpha}{1-\alpha}}}{E\left[\frac{C\beta\lambda\phi}{(1+n)(1+\beta\lambda\phi)}\right]^{\frac{\alpha}{1-\alpha}} + (1-B)} \tag{28}$$

Let us now denote the right-hand side of the above expression by $G(\phi)$. Proving the existence of an equilibrium amounts to find a fixed point of $G(\phi)$, that is, a value of ϕ such that $G(\phi) = \phi$. It is clear that 0 is such a fixed point. But we would like to show that, provided $\alpha < 1/2$, at least one other fixed point

- corresponding to a non-trivial equilibrium - must exist. For that purpose, we shall show that the function $G(\phi)$ satisfies the following properties:[26]

(i) $G(0) = 0$
(ii) $G'(\phi) \geq 0 \ \forall \phi \in [0, 1]$
(iii) $\lim_{\phi \to 1} G(\phi)/\phi < 1$
(iv) $\lim_{\phi \to 0} G'(\phi) = +\infty$ if $\alpha < 1/2$

Proof:
(i)

$$G(0) = \frac{0}{0 + (1-B)} = 0$$

(ii)

$$G'(\phi) =$$
$$\frac{C\beta\lambda\alpha\left[\frac{C\beta\lambda\phi}{(1+n)(1+\beta\lambda\phi)}\right]^{\frac{1}{-1+\alpha}}(1-B)}{E(1+n)(1+\beta\lambda\phi)^2(1-\alpha)\left[-\frac{1}{E}\left(-E-\left[\frac{C\beta\lambda\phi}{(1+n)(1+\beta\lambda\phi)}\right]^{\frac{\alpha}{-1+\alpha}}+B\left[\frac{C\beta\lambda\phi}{(1+n)(1+\beta\lambda\phi)}\right]^{\frac{\alpha}{-1+\alpha}}\right)\right]^2} \geq 0$$

(iii)

$$\lim_{\phi \to 1} G(\phi)/\phi = \lim_{\phi \to 1}\left(\frac{E\left[\frac{C\beta\lambda\phi}{(1+n)(1+\beta\lambda\phi)}\right]^{\frac{\alpha}{1-\alpha}}}{\phi E\left[\frac{C\beta\lambda\phi}{(1+n)(1+\beta\lambda\phi)}\right]^{\frac{\alpha}{1-\alpha}}+\phi(1-B)}\right) =$$
$$\frac{E\left[\frac{C\beta\lambda}{(1+n)(1+\beta\lambda)}\right]^{\frac{\alpha}{1-\alpha}}}{E\left[\frac{C\beta\lambda}{(1+n)(1+\beta\lambda)}\right]^{\frac{\alpha}{1-\alpha}}+(1-B)} < 1$$

(iv)

$$G'(\phi) = \frac{d}{d\phi}\left[\left(1 + \frac{(1-B)}{E\left[\frac{C\beta\lambda\phi}{(1+n)(1+\beta\lambda\phi)}\right]^{\frac{\alpha}{1-\alpha}}}\right)^{-1}\right]$$
$$= E\left[\frac{C\beta\lambda\phi}{(1+n)(1+\beta\lambda\phi)}\right]^{\frac{2\alpha-1}{1-\alpha}}\frac{\left(\frac{C\beta\lambda}{(1+n)(1+\beta\lambda\phi)^2}\right)(1-B)\alpha}{(1-\alpha)\left(-E\left[\frac{C\beta\lambda\phi}{(1+n)(1+\beta\lambda\phi)}\right]^{\frac{\alpha}{1-\alpha}}-1+B\right)^2}$$

Hence,

$$\lim_{\phi \to 0} G'(\phi) = E\left[C\beta\lambda\frac{0}{(1+n)(1)}\right]^{\frac{2\alpha-1}{1-\alpha}}(1-B)\alpha\frac{C\beta\frac{\lambda}{(1+n)(1)^2}}{(1-\alpha)(-1+B)^2}$$
$$= +\infty \text{ iff } \alpha < 1/2$$

---

[26] This part of the proof follows a method proposed by Chakraborty (2004).

One can now see that, provided $\alpha < 1/2$, $\lim_{\phi \to 0} G'(\phi)$ is infinite, so that, given (i), (ii) and (iii), the function $G(\phi)$ must intersect the 45° line at least once on the $[0, 1]$ interval. Hence, in that case, at least one non-trivial equilibrium must exist. Figure 10 makes appear the crucial role played by condition (iv), which insures us that $G(\phi)$ must be above the 45° line in the neighbourhhood of 0, so that, given that $G(1) < 1$, it is clear that a fixed point must exist.

Figure 10: Fixed-point under $\alpha < 1/2$ ($A = 20$, $\tau = 0.1$, $\beta = 0.99$, $\lambda = 0.8$, $n = 0$, $\gamma = 0.1$, $\delta = 0.3$ and $B = 0.5$)
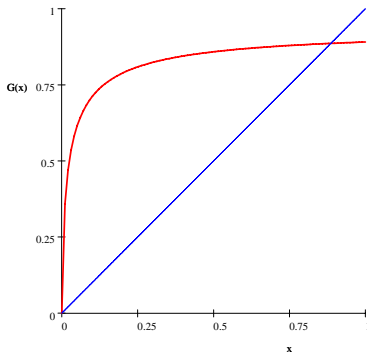
Case 1: $\alpha = 0.4$

Moreover, in the light of the expression of $G'(\phi)$, it is also possible to claim with certainty that such a non-trivial equilibrium must be unique. Indeed, it is straightforward to see that, provided $\alpha < 1/2$, increasing ϕ implies a decrease of the first factor of $G'(\phi)$. Moreover, the numerator of the second factor of $G'(\phi)$ decreases when ϕ increases, while its denominator increases, so that the second factor must also necessarily fall when ϕ increases. Hence, one can conclude from this that $G'(\phi)$ is, under $\alpha < 1/2$, decreasing with ϕ, so that the non-trivial equilibrium is, in that case, unique.

However, if $\alpha \geq 1/2$, the $\lim_{\phi \to 0} G'(\phi) = 0$, so that it is not clear whether a non-trivial equilibrium exists or not. Three sub-cases should be distinguished. These are illustrated on Figure 13.

On the one hand, if the $G(\phi)$ curve lies slightly above the 45° line in the neighbourhood of zero, we know that there must exist a non-trivial equilibrium, because $G(\phi)$ tends to a value strictly lower than unity when ϕ tends to 1, so that the $G(\phi)$ curve must necessarily, in that case, intersect the 45° line on the $[0, 1]$ interval.

On the other hand, if the $G(\phi)$ curve lies below the 45° line in the neighbourhood of zero, it is not clear whether a non-trivial equilibrium exists: two sub-cases should be distinguished. Firstly, it may be the case that the $G(\phi)$ curve intersects the 45° line once from below, and then a second time from above (which must be the case here, because $G(1) < 1$). This case is possible,

because, as we saw above, under $\alpha \geq 1/2$, the $G(\phi)$ curve is not necessarily concave over the entire interval. Secondly, it is also possible that the $G(\phi)$ curve remains below the 45° line over the entire $[0, 1]$ interval, so that no non-trivial equilibrium exists.

Figure 11: Fixed-points under $\alpha \geq 1/2$ ($A = 20$, $\tau = 0.1$, $\beta = 0.99$, $\lambda = 0.8$, $n = 0$, $\gamma = 0.1$, $\delta = 0.3$ and $B = 0.5$)

Case 2a: $\alpha = 0.6$ Case 2b: $\alpha = 0.75$ Case 2c: $\alpha = 0.9$
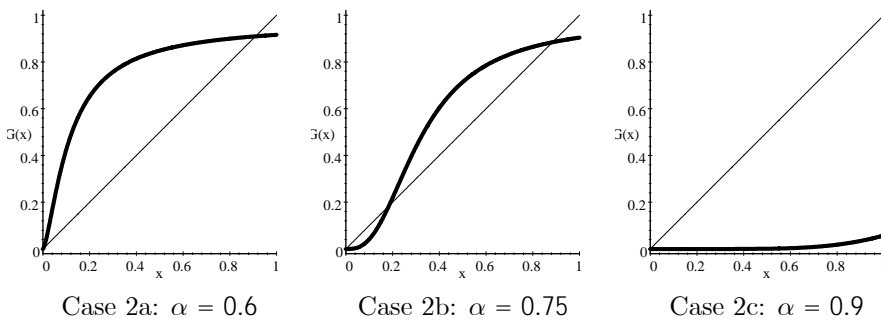
6.2 Appendix B: Stability of steady-states in the basic model

While the discussion of the stability of equilibria carried out in Section 2 in the light of phase diagrams suggested that non-trivial equilibria are likely to be stable (except the intermediate equilibrium), a more formal study of the stability is nonetheless required, because phase diagrams can sometimes simplify the picture significantly.

To discuss the stability of equilibria, let us first notice that the present system is non-linear, so that the conventional analysis of the Jacobian matrix (composed of the first-order derivatives of dynamic equations with respect to state variables) can only inform us on the stability of equilibria provided these are hyperbolic. Actually, if a fixed-point is hyperbolic, the Hartman-Grobman Theorem states that the stability of the linearized system (or its non-stability) implies the local stability of the non-linear system (or its non-stability) (see Medio and Lines, 2001). However, if the fixed-point is not hyperbolic, then the analysis of the linearized system does not allow us to draw any conclusion on the local stability of the non-linear system.

As stated in Medio and Lines (2001), fixed-points are, in discrete-time systems, hyperbolic if none of the eigenvalues of the Jacobian matrix, evaluated at the equilibrium, is equal to 1 in modulo.

To discuss the hyperbolicity of non-trivial equilibria, let us first compute the Jacobian matrix associated with the basic model of Section 2:

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$$J \equiv \begin{pmatrix} \frac{\partial M(k_t, \phi_t^{t-1})}{\partial k_t} & \frac{\partial M(k_t, \phi_t^{t-1})}{\partial \phi_t^{t-1}} \\ \frac{\partial N(k_t, \phi_t^{t-1})}{\partial k_t} & \frac{\partial N(k_t, \phi_t^{t-1})}{\partial \phi_t^{t-1}} \end{pmatrix} \tag{29}$$

where $M(k_t, \phi_t^{t-1})$ and $N(k_t, \phi_t^{t-1})$ denote:

$$M(k_t, \phi_t^{t-1}) \equiv \frac{(1-\alpha)(1-\tau)}{1+n} A k_t^\alpha \frac{\beta \lambda \phi_t^{t-1}}{1 + \beta \lambda \phi_t^{t-1}} \tag{30}$$

$$N(k_t, \phi_t^{t-1}) \equiv \frac{B\phi_t^{t-1} + (1-\alpha)Ak_t^\alpha(\gamma(1-\tau) + \delta\tau)}{1 + (1-\alpha)Ak_t^\alpha(\gamma(1-\tau) + \delta\tau)} \tag{31}$$

The relevant first-order derivatives are:

$$\frac{\partial M(k_t, \phi_t^{t-1})}{\partial k_t} = (1-\alpha)(1-\tau) A k_t^{\alpha-1} \alpha \beta \lambda \frac{\phi_t^{t-1}}{(1+n)\left(1 + \beta \lambda \phi_t^{t-1}\right)} > 0 \tag{32}$$

$$\frac{\partial M(k_t, \phi_t^{t-1})}{\partial \phi_t^{t-1}} = (1-\alpha)(1-\tau) A k_t^\alpha \beta \frac{\lambda}{(1+n)\left(1 + \beta \lambda \phi_t^{t-1}\right)^2} > 0 \tag{33}$$

$$\frac{\partial N(k_t, \phi_t^{t-1})}{\partial k_t} = \frac{(1-\alpha)A\alpha((1-\tau)\gamma + \delta\tau)k_t^\alpha(1-B\phi)}{k_t\left(-1 - Ak_t^\alpha((1-\tau)\gamma + \delta\tau)(1-\alpha)\right)^2} > 0 \tag{34}$$

$$\frac{\partial N(k_t, \phi_t^{t-1})}{\partial \phi_t^{t-1}} = \left[\frac{B}{1 + Ak_t^\alpha((1-\tau)\gamma + \delta\tau)(1-\alpha)}\right] > 0 \tag{35}$$

An equilibrium is hyperbolic if no eigenvalue of the Jacobian matrix, evaluated at the equilibrium, is equal to 1 in modulo. A sufficient condition for this hyperbolicity condition is that the trace of the Jacobian matrix (i.e. the sum of eigenvalues) is positive but strictly lower than 1, and that the determinant of the Jacobian matrix (i.e. the product of eigenvalues) is positive but strictly lower than one.

Actually, it is easy to see that those conditions are also sufficient conditions for the local stability of the equilibrium, which requires that all eigenvalues of the Jacobian matrix are strictly lower than 1 in modulo.

These two conditions imply that, at the equilibrium (k^*, ϕ^*):

$$0 < (1-\alpha)(1-\tau)Ak^{*\alpha-1}\alpha\beta\lambda\frac{\phi^*}{(1+n)(1+\beta\lambda\phi^*)} + \left[\frac{B}{1+Ak^{*\alpha}(\gamma(1-\tau)+\delta\tau)(1-\alpha)}\right] < 1$$

$$(*)$$

and

$$0 < \left[\frac{(1-\alpha)(1-\tau)Ak^{*\alpha-1}\alpha\beta\lambda\phi^*B}{(1+n)(1+\beta\lambda\phi^*)[1+Ak^{*\alpha}(\gamma(1-\tau)+\delta\tau)(1-\alpha)]}\right] - $$
$$\left[\frac{(1-\alpha)(\gamma(1-\tau)+\delta\tau)\alpha Ak^{*\alpha}(1-B\phi^*)(1-\alpha)(1-\tau)Ak^{*\alpha}\beta\lambda}{k^*(-1-Ak^{*\alpha}(\gamma(1-\tau)+\delta\tau)(1-\alpha))^2(1+n)(1+\beta\lambda\phi^*)^2}\right] < 1 \ (**)$$

Regarding condition (*), the positivity of the sum in question is obvious. Hence (*) can be reformulated as:

$$(1-\alpha)(1-\tau)Ak^{*\alpha-1}\alpha\beta\lambda\frac{\phi^*}{(1+n)(1+\beta\lambda\phi^*)} < 1 - \frac{B}{1+Ak^{*\alpha}(\gamma(1-\tau)+\delta\tau)(1-\alpha)}$$

which is likely to be satisfied when the equilibrium is reached at a high level of capital.

Regarding (**), the positivity constraint can be rewritten as:

$$\left[\frac{(1-\alpha)(1-\tau)Ak^{*\alpha-1}\alpha\beta\lambda\phi^*}{(1+n)(1+\beta\lambda\phi^*)}\frac{B}{1+Ak^{*\alpha}(\gamma(1-\tau)+\delta\tau)(1-\alpha)}\right] > $$
$$\left[\frac{(1-\alpha)(\gamma(1-\tau)+\delta\tau)\alpha Ak^{*\alpha}(1-B\phi^*)}{k^*(-1-Ak^{*\alpha}(\gamma(1-\tau)+\delta\tau)(1-\alpha))^2}\frac{(1-\alpha)(1-\tau)Ak^{*\alpha}\beta\lambda}{(1+n)(1+\beta\lambda\phi^*)^2}\right]$$

or, in short,

$$B\phi^* > \left[\frac{Ak^{*\alpha}(\gamma(1-\tau)+\delta\tau)(1-\alpha)}{(1+Ak^{*\alpha}(\gamma(1-\tau)+\delta\tau)(1-\alpha))}\frac{(1-B\phi^*)}{(1+\beta\lambda\phi^*)}\right]$$

which is, here again, a plausible condition if the equilibrium considered is characterized by sufficiently high ϕ. Actually, given that the two factors on the right-hand side are smaller than unity, the above condition is likely to be satisfied provided B is sufficiently large.

Regarding the second part of condition (**), it implies that:

$$(1-\alpha)(1-\tau)Ak^{*\alpha-1}\alpha\beta\lambda\phi^*B < $$

$$\frac{k^*(1+Ak^{*\alpha}(\gamma(1-\tau)+\delta\tau)(1-\alpha))^2(1+n)(1+\beta\lambda\phi^*)^2+(1-\alpha)^2(\gamma(1-\tau)+\delta\tau)(1-\tau)\alpha A^2k^{*2\alpha}(1-B\phi^*)\beta\lambda}{k^*(1+Ak^{*\alpha}(\gamma(1-\tau)+\delta\tau)(1-\alpha))(1+\beta\lambda\phi^*)}$$

Given that all factors on the left-hand side are - except A - smaller than 1 by hypothesis, and that the right-hand side is clearly higher than 1, this condition is likely to be satisfied, especially when k^* is high.

To conclude, the present discussion of the stability of the equilibrium suggests that this stability cannot be fully guaranteed in all cases. However, as it was shown, the equilibrium is likely to be stable when it corresponds to a sufficiently high combination of k^* and ϕ^*. The reason for this is simple: being distant from the other equilibrium - i.e. the (0,0) point - insures stability, whereas an equilibrium that is close to the (0,0) is likely to be unstable.

6.3 Appendix C: Existence of steady-states in the extended model

In order to derive the conditions guaranteeing the existence of a non-trivial steady-state, we shall follow here the technique used in Appendix A.

Actually, one can rewrite the $\phi\phi$ locus as:

$$\phi = \frac{k^\alpha V}{k^\alpha V + 1 - B} \tag{36}$$

where $V \equiv A(1-\alpha)(\gamma(1-\tau) + \delta\psi\tau)$. Given the assumptions $A > 0$, $0 < \alpha < 1$ and $\gamma(1-\tau) + \delta\psi\tau > 0$, it follows that V must be strictly positive.

In a similar way, the $\lambda\lambda$ locus can be written as:

$$\lambda = \frac{k^\alpha W}{k^\alpha W + 1 - R} \tag{37}$$

where $W \equiv A(1-\alpha)(\varepsilon(1-\tau) + \upsilon(1-\psi)\tau)$. Here again, $W > 0$.

Then, substituting these two expressions in the kk locus yields:

$$k = \left(\frac{A\beta(1-\alpha)(1-\tau)\left(\frac{k^\alpha V}{k^\alpha V + 1 - B}\right)\left(\frac{k^\alpha W}{k^\alpha W + 1 - R}\right)}{(1+n)\left(1 + \beta\left(\frac{k^\alpha V}{k^\alpha V + 1 - B}\right)\left(\frac{k^\alpha W}{k^\alpha W + 1 - R}\right)\right)} \right)^{\frac{1}{1-\alpha}} \tag{38}$$

If we denote the right-hand side as the function $H(k)$, the question of the existence of an equilibrium becomes the search for a fixed point of $H(k)$. It is clear that 0 is a fixed point. But we would like to derive conditions guaranteeing the existence of another fixed point.

For that purpose, we shall follow the method proposed in the Appendix A and show that the function $H(k)$ satisfies the properties (i)-(iv), that is:

(i) $H(0) = 0$
(ii) $H'(k) \geq 0 \ \forall k \in [0, \bar{k}]$
(iii) $\lim_{k \to \bar{k}} H(k)/k < 1$
(iv) $\lim_{k \to 0} H'(k) = +\infty$ if $\alpha < \alpha^*$.

(i)

$$H(0) = \left(\frac{A\beta(1-\alpha)(1-\tau)0}{(1+n)} \right)^{\frac{1}{1-\alpha}} = 0$$

(ii) To show this, we shall assume that there exists an upper bound steady-state k, denoted by $\bar{k}$, i.e. a value of k that the steady-state capital never exceeds. Given that the postulated Cobb-Douglas production function satisfies the Inada conditions, there can be no doubt that such an upper bound exists. Moreover, that upper bound is independent from ϕ and λ. Throughout this Appendix, we shall take $\bar{k} = \left(\frac{A\beta(1-\alpha)(1-\tau)}{(1+n)(1+\beta)} \right)^{\frac{1}{1-\alpha}}$

In the interval $[0, \bar{k}]$, the first-order derivative of H with respect to k has, under mield conditions, the desired sign. Indeed, a positive $H'(k)$ implies:

$$k^{2\alpha}(1 - \frac{(VW)^{\alpha+1}}{\beta}) \leq \frac{(VW)^{\alpha} k^{\alpha} V(1-R)}{\beta} + \frac{(VW)^{\alpha} k^{\alpha} W(1-B)}{\beta} + \frac{(1-B)(1-R)}{\beta} + 1 \tag{39}$$

which is a mild condition.

(iii) When k tends to $\bar{k}$, the $\lim_{k \to \bar{k}} H(k)/k$ is:

$$\lim_{k \to \bar{k}} H(k)/k = \frac{\left(\frac{A\beta(1-\alpha)(1-\tau)\left(\lim_{k \to \bar{k}} \phi\right)\left(\lim_{k \to \bar{k}} \lambda\right)}{(1+n)\left(1+\beta\left(\lim_{k \to \bar{k}} \phi\right)\left(\lim_{k \to \bar{k}} \lambda\right)\right)} \right)^{\frac{1}{1-\alpha}}}{\left(\frac{A\beta(1-\alpha)(1-\tau)}{(1+n)(1+\beta)} \right)^{\frac{1}{1-\alpha}}} < 1$$

given that ϕ and λ are bounded by 1 (i.e. their values when $k \to +\infty$), the above ratio remains inferior to 1 as required.

(iv) The $\lim_{k \to 0} H'(k)$ is:

$$\frac{C^{\frac{1}{1-\alpha}}}{1-\alpha} \frac{\left(\left(k^{\frac{3\alpha-1}{1-\alpha}} \right) [(VW)^{\alpha}(1-B)(1-R)]^{\frac{1}{1-\alpha}} (\alpha 2Z) \right) + \left(\beta(VW)^{\frac{1}{1-\alpha}} [(1-B)(1-R)]^{\frac{\alpha}{1-\alpha}} (\alpha 2Z) \right) \left(k^{\frac{3\alpha-1}{1-\alpha}} \left(1 - k^{2\alpha} \right) \right)}{(1+n)^{\frac{1-2\alpha}{1-\alpha}} (1-B)^{\frac{3-\alpha}{1-\alpha}} (1-R)^{\frac{3-\alpha}{1-\alpha}}}$$

where $Z \equiv V(1-B)W(1-R)$. It is straightforward to see that, when k tends to zero, $H'(k)$ tends to $+\infty$ provided $\alpha < \alpha^* = 1/3$.

By the same rationale as in the basic case, it appears that, under $\alpha < 1/3$, a non-trivial equilibrium must exist: the function $H(k)$ must intersect the 45° line at least once on the interval $[0, \bar{k}]$. However, for $\alpha \geq 1/3$, one cannot be sure that a non-trivial equilibrium exists.

Regarding the unicity issue, one can see, from the above expression, that, under $\alpha < 1/3$, $H''(k)$ is strictly negative, so that H is concave, which implies that the non-trivial equilibrium must be, under that condition, unique.

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